The Final Round of the Mathematics Competition of Seventh Graders of Helsinki on 21 April, 2016
Solutions

1. Pascal's triangle is defined by first writing 1, and then under it two ones, and each new row is one number longer than the previous ones, it begins and ends with a one, and the numbers in between are each obtained by adding together the two numbers above it:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

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How many times does the number 10 appear in Pascal's triangle?

Solution. The beginning of Pascal's triangle looks as follows:

$$\begin{array}{ccccccccccccccccccccc} &&&&&&&&&&1&&&&&&&&&& \\ &&&&&&&&&1&&1&&&&&&&&& \\ &&&&&&&&1&&2&&1&&&&&&&& \\ &&&&&&&1&&3&&3&&1&&&&&&& \\ &&&&&&1&&4&&6&&4&&1&&&&&& \\ &&&&&1&&5&&10&&10&&5&&1&&&&& \\ &&&&1&&6&&\dots&&\dots&&\dots&&6&&1&&&& \\ &&&1&&7&&\dots&&\dots&&\dots&&\dots&&7&&1&&& \\ &&1&&8&&\dots&&\dots&&\dots&&\dots&&\dots&&8&&1&& \\ &1&&9&&\dots&&\dots&&\dots&&\dots&&\dots&&\dots&&9&&1& \\ 1&&10&&\dots&&\dots&&\dots&&\dots&&\dots&&\dots&&\dots&&10&&1 \end{array}$$

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From this onwards, everything except for the ones on the borders is greater than ten. Thus the number 10 appears exactly four times.

2. What is the one hundreth decimal in the number 3/7?

Solution. By performing the division 3/7 we obtain

$$0{,}4285714\ldots$$

Since in this long division each decimal depends only on the previous one, we see that the decimals repeat themselves in groups of six. In particular, after the decimal point every sixth decimal is 1. Therefore the 96th decimal is 1, the 97th decimal is 4, the 98th decimal is 2, the 99th decimal is 8, and finally the 100th decimal is 5.

3. We know that

$$1^3 + 2^3 + 3^3 + 4^3 + \ldots + 10^3 = 3025,$$

and that

$$1^3 + 2^3 + 3^3 + 4^3 + \ldots + 20^3 = 44100.$$

What is

$$1^3 + 3^3 + 5^3 + 7^3 + \ldots + 19^3?$$

Solution. The last sum can be written in terms of the two previous ones:

$$\begin{aligned} &1^3 + 3^3 + 5^3 + 7^3 + \ldots + 19^3 \\ &= 1^3 + 2^3 + 3^3 + 4^3 + \ldots + 20^3 - \left(2^3 + 4^3 + 6^3 + \ldots + 20^3\right) \\ &= 44100 - 8\left(1^3 + 2^3 + 3^3 + \ldots + 10^3\right) \\ &= 44100 - 8 \cdot 3025 = 44100 - 24200 = 19900. \end{aligned}$$

4. Let us consider sequences formed from the symbols $\heartsuit$ and $\diamondsuit$. We are allowed to perform three sorts of operations on our sequences:

- We can always erase two consecutive $\heartsuit$ symbols;
- we can always replace the consecutive symbols $\heartsuit\diamondsuit\heartsuit$ by the consecutive symbols $\diamondsuit\diamondsuit$; and
- we can always replace the consecutive symbols $\diamondsuit\diamondsuit$ by the consecutive symbols $\heartsuit\diamondsuit\heartsuit$.

Determine how these operations can be used to turn the sequence $\diamondsuit\diamondsuit\diamondsuit\diamondsuit\diamondsuit\diamondsuit\diamondsuit$ into the sequence $\diamondsuit$.

Solution. We can operate as follows:

$$\diamondsuit\diamondsuit\diamondsuit\diamondsuit\diamondsuit\diamondsuit\diamondsuit \longmapsto \heartsuit\diamondsuit\heartsuit\heartsuit\diamondsuit\heartsuit\diamondsuit\heartsuit\diamondsuit\heartsuit \longmapsto \heartsuit\diamondsuit\diamondsuit\heartsuit\diamondsuit\heartsuit\diamondsuit\heartsuit \longmapsto \heartsuit\heartsuit\diamondsuit\heartsuit\heartsuit\diamondsuit\heartsuit\diamondsuit\heartsuit$$
$$\longmapsto \diamondsuit\diamondsuit\heartsuit\diamondsuit\heartsuit \longmapsto \heartsuit\diamondsuit\heartsuit\heartsuit\diamondsuit\heartsuit \longmapsto \heartsuit\diamondsuit\diamondsuit\heartsuit \longmapsto \heartsuit\heartsuit\diamondsuit\heartsuit\heartsuit \longmapsto \diamondsuit.$$

5. We know that when a positive integer is divided by five, the remainder is two. What are the possible remainders, when the number in question is divided by seven?

Solution. The remainder when dividing by seven is always either 0, 1, 2, 3, 4, 5 or 6. We show that each of these is possible: namely, for the numbers 7, 12, 17, 22, 27, 32 and 37, we have

$$7 = 1 \cdot 7 + 0, \quad 12 = 1 \cdot 7 + 5, \quad 17 = 2 \cdot 7 + 3, \quad 22 = 3 \cdot 7 + 1,$$
$$27 = 3 \cdot 7 + 6, \quad 32 = 4 \cdot 7 + 4, \quad \text{and} \quad 37 = 5 \cdot 7 + 2.$$

6. There are 100 lines in the plane. Some of them may be parallel with one or more other lines, but no three of them intersect at the same point. Is it possible that the lines have alltogether exactly 2016 points of intersection?

Solution. Yes, it is possible: let us draw first 72 mutually parallel lines, and then 28 mutually parallel lines, which are not parallel to the previous lines. In this way, we obtain $72 \cdot 28 = 2016$ points of intersection.