The Finnish High School Mathematics Competition 2009–2010 and the Nordic Mathematical Competition 2010

The Finnish High School competition has two rounds. The first round has three divisions, Basic, Intermediate and Open. Participation in the Basic and Intermediate divisions depends on the age of the student, and the participants basically come from the first and second year students. The Open Division has no age limit, but most participants are third year students. The first round took part on October 29, 2009, at schools. The time allowed was 100 minutes. The problems were as follows:

Basic Division

B1. Berries were sold in boxes, and the price depended on the kind of berries in the box. One had to pay 8 euros for two boxes of raspberries, two boxes of currants and one box of blackberries, 7.50 euros for one box of raspberries, three boxes of currants and one box of blackberries and 7 euros for two boxes of raspberries together with three boxes of blackberries. What was the price of three boxes of raspberries, two boxes of curranta and three boxes of blackberries?

B2. Fill in the square below in such a way that all the numbers 1, 2, . . . , 16 appear and the sum of the numbers in each row and column is the same. Find all the ways to complete the square.

4			
9			8
7	2	10	

B3. Two circular discs of radius r are placed on the bottom of a box. The bottom is a square of side a. Determine the smallest possible a.

B4. Find all ways of expressing 2009 as the difference of the squares (second poweres) of two positive integers.

Intermediate Division

I1. The same as B2.

I2. The altitude and base of an isosceles triangle are equal. A circle is drawn with the altitude a diameter. In which ratio does the circle divide the sides of the triangle?

I3. Each of five equally strong players plays a game with all the others. No game ends in draw and the probability of a win of each player is $\frac{1}{2}$. The games are independent of each other. Find the probability of each player winning exactly two games.

I4. Find all ways of expressing 2009 as the difference of cubes (third powers) of positive integers.

Open Division

O1. The legs of a right triangle are 10 and 24. A circle is drawn with its centre on the longer leg such that it is tangent to the hypotenuse and shorter leg. Find the radius of the circle.

O2. The lengths of the sides of a triangle form a geometric procession with ratio q. Prove that $\sqrt{5} - 1 < 2q < \sqrt{5} + 1$.

O3. The same as I4.

O4. Prove that 10 Finns can make 30 phone calls to 10 Swedes in such a way that
(i) no one calls anyone twice;
(ii) no two Finns make all the four possible calls to any two Swedes.

Final Round

The final round took place in Helsinki on January 29, 2010. The top 15 of the Open Division, the top 5 of the Intermediate Division and the winner of the Basic Division participated. The time allowed was three hours. The problems were as follows:

F1. Prove that the sum of the squares of the medians of a right triangle equals $\frac{3}{4}$ of the sum of the squares of the sides of the triangle.

F2. Find the smallest n, for which the number $n!$ has at least 2010 different divisors.

F3. Let $P(x)$ be a polynomial with integer coefficients. Assume 1997 and 2010 are roots of P. Assume further, that $|P(2005)| < 10$. Find the possible integer values of $P(2005)$.

F4. An even number n of football teams play a single series, i.e. each team plays exactly once against every other team. Prove that the matches can be grouped into $n - 1$ rounds in such a way that each team plays once on every round.

F5. Let S be a set of points in the plane. We say that a point P is visible from a point A, if all points on the segment AP belong to the set S, and that the set S is visible from a point A, if evey point of S is visible from A. Assume that S is visible from all vertices of a triangle ABC. Show that S is visible from all points of the triangle ABC.

The Nordic Competition

The 24th Nordic Mathematical Competition took place on April 23, 2010, with participants from Denmark, Iceland, Norway, Sweden and Finland workin at their schools. The Competition was administered by Finland. The time allowed was four hours and the problems were suggested by Iceland, Denmark, Sweden and Norway.

N1. A function $f : \mathbb{Z}_+ \to \mathbb{Z}_+$, where $\mathbb{Z}_+$ is the set of positive integers, is non-decreasing and satisfies $f(mn) = f(m)f(n)$ for all realtively prime positive integers m and n. Prove that $f(8)f(13) \geq (f(10))^2$.

N2. Three circles Γ_A, Γ_B and Γ_C share a common point of intersection O. The other common point of Γ_A and Γ_B is C, that of Γ_A and Γ_C is B and that of Γ_B and Γ_C is A. The line AO intersects the circle Γ_A in the point $X \neq O$. Similarly, the line BO intersects the circle Γ_B in the point $Y \neq O$, and the line CO intersects the circle Γ_C in the point $Z \neq O$. Show that

$$\frac{|AY|\,|BZ|\,|CX|}{|AZ|\,|BX|\,|CY|} = 1.$$

N3. Laura has 2010 lamps connected with 2010 buttons in front of her. For each button, she wants to know the corresponding lamp. In order to do this, she observes which lamps are lit when Richard presses a selection of buttons. (Not pressing anything is also a possible selection.) Richard always presses the buttons simultaneously, so the lamps are lit simultaneously, too.

a) If Richard chooses the buttons to be pressed, what is the maximum number of different combinations of buttons he can press until Laura can assign the buttons to the lamps correctly?

b) Supposing that Laura will choose the combinations of buttons to be pressed, what is the minimum number of attempts she has to do until she is able to associate the buttons with the lamps in a correct way?

N4. A positive integer is called *simple* if its ordinary decimal representation consists entirely of zeroes and ones. Find the least positive integer k such that each positive integer n can be written as $n = a_1 \pm a_2 \pm a_3 \pm \ldots \pm a_k$ where $a_1, \ldots, a_k$ are simple.

Solutions

B1. Let v, h and m be the price of a box of raspberries, currants and blackberries, respectively. Then v, h and m satisfy

$$\begin{cases} 2v + 2h + m = 8 \\ v + 3h + m = 7{,}5 \\ 2v \phantom{{}+3h} + 3m = 7, \end{cases}$$

Adding the equations, one obtains $5(v + h + m) = 22{,}5$ or $v + h + m = 4{,}5$. This and the second equation yield $2h = 7{,}5 - (h + v + m) = 7{,}5 - 4{,}5 = 3$, or $h = 1{,}5$. The second equation now gives $3v + 2h + 3m = 3(v + h + m) - h = 3 \cdot 4{,}5 - 1{,}5 = 12$.

B2, I1. The sum of the numbers in the grid is

$$1 + 2 + \cdots + 16 = \frac{16 \cdot 17}{2} = 8 \cdot 17 = 4 \cdot 34,$$

so the row and column sums are 34. The lowest square on the leftmost column must contain 14 and the fourth square on the third row must contain 15. The sum of the two empty squares in the fourth column have the sum 11. Of all the pairs having sum 11, only (5, 6) is available. One has to check the two ways of placing 5 and 6 in the free squares. If 5 is in the first row, a series of eliminations of impossible combinations leads to the placement

4	13	12	5
9	16	1	8
7	2	10	15
14	3	11	6

If we put 6 in the first row, a similar chain of argument leads to the necessary placement

4	13	11	6
9	16	1	8
7	2	10	15
14	3	12	5

So there are two ways of completing the square.

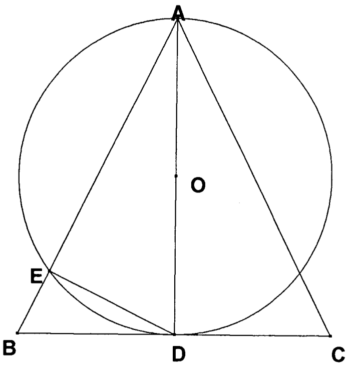
B3. Let the line joining the centers of the discs form the angle α with one pairs of sides of the square, say the horizontal one. The distance of the vertical sides must be at least $r + 2r\cos\alpha + r = 2r(1+\cos\alpha)$ and the distance of the horizontal sides must be at least $r+2r\sin\alpha+r = 2r(1+sin\alpha)$. So a has to be at least the larger of the numbers $2r(1+\cos\alpha)$, $2r(1+\sin\alpha)$. If $\alpha = 45°$, both numbers equal $2r\left(1+\frac{\sqrt{2}}{2}\right)$. If $\alpha < 45°$, $\cos\alpha > \cos 45°$ and if $\alpha > 45°$, $sin\alpha > \sin 45°$. The larger distance is minimal for $\alpha = 45°$. The smallest possible a equals $2r\left(1+\frac{\sqrt{2}}{2}\right)$ (in the case of discs touching each other and the sides of the square, of course).

B4. We look for positive integers x, y satisfying $2009 = x^2 - y^2 = (x+y)(x-y)$. As $x+y$ ja $x-y$ must be factors of $2009 = 7 \cdot 287 = 7^2 \cdot 41$, and because $x+y > x-y$, x and y have to satisfy one of the systems

$$\begin{cases} x+y = 2009 \\ x-y = 1 \end{cases}, \quad \begin{cases} x+y = 287 \\ x-y = 7 \end{cases}, \quad \begin{cases} x+y = 49 \\ x-y = 41 \end{cases}.$$

The systems have solutions $(x,\, y) = (1005,\, 1004)$, $(147,\, 140)$ and $(45,\, 4)$.

I2. The problem only makes sense if the altitude mentioned in the problem is the one dropped on the base. Let $AB = AC$ in ABC, let $BC = 2r$ and let $AD = 2r$ be the altitude. Let O be the midpoint of AD and let the circle with center O and radius r intersect AB at E. Choose $EB = 1$ and let $AE = x$. Then $\angle AED = 90°$, and $DE = h$ is the altitude of ABD. By well-known properties of right triangles, $h^2 = AE \cdot EB = x$. From BDE one obtains $x = h^2 = r^2 - 1$ or $r^2 = 1 + x$. From AED one obtains $x^2 + h^2 = 4r^2$. So $x^2 + x = 4(x + 1)$ or $x^2 - 3x - 4 = 0$. The only positive solution of this quadratic is $x = 4$. So the division ratio is $4 : 1$.



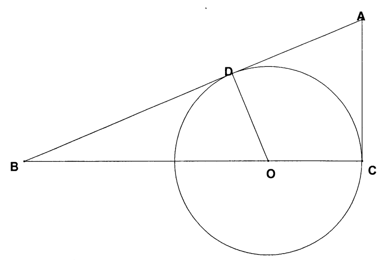
I3. Each player plays four games and there are altogether $\binom{5}{2} = 10$ games. Let P_1 be one of the players. The probability of P_1 winning exactly two games is $\binom{4}{2}\left(\frac{1}{2}\right)^4 = \frac{6}{16} = \frac{3}{8}$. Assume that P_1 wins against P_2 and P_3 and loses against P_4 and P_5. Further assume that P_2 wins against P_3 and P_4 wins against P_5. There are still four games to consider. To reach the desired conclusion, P_3 has to win P_4 and P_5. Now P_5 has to win P_2 and P_4 has to win P_4. The probability of these four games having these results in $\frac{1}{2^4} = \frac{1}{16}$. Thus the probability of exactly two wins for each player is $\frac{3}{8} \cdot \frac{1}{16} = \frac{3}{128}$.

I4, O3. The equation to be solved is $2009 = x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ ja $2009 = 7^2 \cdot 41 = 7 \cdot 287$. So the integers $x > y$, satisfy one of the systems

$$\begin{cases} x - y = 1 \\ x^2 + xy + y^2 = 2009 \end{cases}, \quad \begin{cases} x - y = 7 \\ x^2 + xy + y^2 = 287 \end{cases}, \quad \begin{cases} x - y = 41 \\ x^2 + xy + y^2 = 49 \end{cases}.$$

The system leads to $x^2 + x(x-1) + (x-1)^2 = 2009$ or $3x^2 - 3x = 2008$. Since 3 is not a divisor of 2008, there are no solutions. The second system leads to $x^2 + x(x - 7) + (x - 7)^2 = 287$ or $3x^2 - 21x + 49 = 7 \cdot 41$. So x^2 is divisible by 7. Then x^2 and $21x$ are divisible by 49. As the right-hand side of the equation is not divisible by 49, there are no solutions. If x is an intger solution of the last system, then $x \geq 41$ ja $x^2 + xy + y^2 \geq 41^2 > 49$. So there are no solutions to the problem.

O1. Let C be the vertex of the right angle in ABC, and let $AC = 10$, $BC = 24$. Then $AB^2 = 4 \cdot (5^2 + 12^2) = 4 \cdot 169 = 26^2$, and $AB = 26$. Let O be a point on BC and let the circle Γ with center O and radius r touch AC and AB. Since $BC \perp AC$, Γ touches AC at C. Let D be the common point of Γ and AB. So $OD \perp AB$. By the well-known property of tangents, $AD = AC = 10$. So $BD = 26 - 10 = 16$. The right



triangles ABC and BOD are similar. Consequently $\dfrac{r}{BD} = \dfrac{AC}{BC} = \dfrac{5}{12}$ and $r = \dfrac{16 \cdot 5}{12} = \dfrac{20}{3} = 6\,\dfrac{2}{3}$.

O2. The sides of the triangle are a, qa ja q^2a. Assume that $q \geq 1$. By the triangle inequality, $aq^2 < a + aq$ or $q^2 - q - 1 < 0$. Since the equality turns into an equation when $q = \dfrac{1}{2}(1 \pm \sqrt{5})$, the inequality holds for $1 \leq q < \dfrac{1}{2}(1 + \sqrt{5})$. So $2 \leq 2q < 1 + \sqrt{5}$. Then assume $0 < q < 1$. Now a is the longest side, and $a < aq + aq^2$ or $q^2 + q - 1 > 0$. As above, we conclude that the inequality holds for $\dfrac{1}{2}(-1 + \sqrt{5}) < q < 1$ or $-1 + \sqrt{5} < 2q < 2$. The numer q satisfies the double inequality of the problem.

O4. For a solution, it suffices to describe one way of arranging the calls so that the condiotion of the problem are met. Let S_0, S_2, $\ldots$, S_9 be the Finns and R_1, R_2, $\ldots$, R_9 the Swedes. Make the Finn S_i call the Swedes R_i, R_{i+1} and R_{i+3}, reading the indices mod 10. There are $3 \cdot 10 = 30$ calls and no Finn calls the same Swede twice, so cindition (i) is fulfilled. To see that condition (ii) holds, inspect the matrix shoving all the calls:

$$
\begin{array}{c}
 \\ R_0 \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \\ R_9
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc} S_0 & S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 & S_9 \end{array} \\
\left(\begin{array}{cccccccccc}
x & x & & x & & & & & & \\
 & x & x & & x & & & & & \\
 & & x & x & & x & & & & \\
 & & & x & x & & x & & & \\
 & & & & x & x & & x & & \\
 & & & & & x & x & & x & \\
 & & & & & & x & x & & x \\
x & & & & & & & x & x & \\
 & x & & & & & & & x & x \\
x & & x & & & & & & & x
\end{array}\right)
\end{array}.
$$

If some two finns S_i and S_k would make all four possible calls to some two Swedes R_m and R_n, there would be an x in all the vertices of the rectangle determined by the rows m and n and the columns i and k. An inspection of the matrix shows that this is not the case.

F1. The assumption that ABC is a right triangle is not needed. Let ABC be a triangle, $BC = a$, $CA = b$ and $AB = c$. Let D, E and F be the midpoints of BC, CA and AB, and let $AD = m_a$, $BE = m_b$ and $CF = m_c$ be the medians. By the well known parallelogram theorem (easy to derive from the theorem of Pythagoras) the sum of the squares of the diameters a parallellogram equals the sum of its squares of the sides.There are three possibilities to extend the triangle into a parallelogram: withs sides a and b and diameters c and $2m_c$, and the two others obtained by circular permutations. Applying the parallellogram theorem to each of these parallelograms gives $c^2 + 4m_c^2 = 2(a^2 + b^2)$, $a^2 + 4m_a^2 = 2(b^2 + c^2)$ and $b^2 + 4m_b^2 = 2(a^2 + b^2)$. The conclusion follows when the equations are added.

F2. If the prime factor decomposition of n is $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, then the number of divisors of n equals $d(n) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$. For $m < n$, every divisor of $m!$

divides $n!$, but $n!$ has divisors not dividing $m!$ (e.g $n!$). So $d(n!)$ is a strictly increasing function of n. Now 16! has $8+4+2+1 = 15$ 2's, $5+1 = 6$ 3's 3 5's , 2 7's one 11 and one 13 as divisors. So there are $16 \cdot 7 \cdot 4 \cdot 3 \cdot 2 \cdot 2 = 2^8 \cdot 21 = 21 \cdot 256 > 5000$ factors. 15! only has 11 2's $d(15!)$ is $\frac{12}{16} = \frac{3}{4}$ times the $d(16!)$. So $d(15!) > 3000$. 14! has one less of 3's and 5's, i.e. $d(14!) = 12 \cdot 5 \cdot 3 \cdot 3 \cdot 2 \cdot 2 = 60 \cdot 36 = 2160 > 2010$. Going down to 13!, there will be one seven instead of two, which means that $d(13!)$ is less than $\frac{2}{3}$ of $d(14!)$, or clearly less than 2000. So $n = 14$ is the answer.

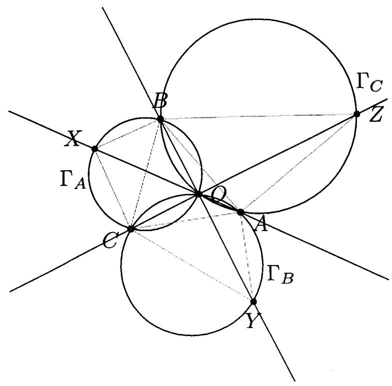
F3. If $P(x_0) = 0$, then $P(x) = (x - x_0)Q(x)$. If P has integer coefficients and x_0 is an integer, then Q also has integer coefficients. [Proof: If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and $Q(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0$, then $a_n = b_n$, $a_n = b_{n-1}$, $a_{n-1} = b_{n-2} - x_0 b_{n-1}$, $a_{n-2} = b_{n-3} - x_0 b_{n-2}$, $\ldots$, $a_1 = b_0 - x_0 b_1$. Solving b_{n-1}, b_{n-2}, $\ldots$, b_0, in turn from these equations, one sees that all are integers.] So $P(x) = (x - 1997)(x - 2010)Q(x)$, where Q is a polynomial with integer coefficients. In particular, $|P(2005)| = |2005 - 1997| \cdot |2005 - 2010| \cdot |Q(2005)| = 40|Q(2005)|$. If the integer $Q(2005)$ is non-zero, then $|P(2005)| \geq 40 > 10$. So $Q(2005) = 0$ and $P(2005) = 0$.

F4. Label the teams as $1, 2, \ldots, n$. On round i, let the team x play against y, $1 \leq y \leq n-1$ such that $x + y + i \equiv 0 \mod n - 1$, and let the team x for which $2x + i$ is a multiple of $n - 1$ play against n. For any $x \neq y$, there is exactly one of the $n - 1$ consecutive numbers $x + y + 1, \ldots, x + y + n - 1$ is a multiple of $n - 1$, so there is a round when x and $y < n$ play against each other. Also, among the numbers $2x + 1, \ldots, 2x + n - 1$ exactly one is a multiple of $n - 1$, so there is a round when x plays against n. On the other hand, for a fixed i, $1 \leq i \leq n - 1$, there is only one x, $1 \leq x \leq n - 1$ such that $2x + i \equiv 0 \mod n - 1$ and for x such that $2x + i$ is no a multiple of $n - 1$ there is just one y, $1 \leq y \leq n - 1$ such that $x + y + i \equiv 0 \mod n - 1$. So on every round every team has an opponent.

F5. Assume S is visible from P and Q and let X be a point on the segment PQ. If S is not visible from X, there is a point in Y in S such that the segment XY is not in S. So let Z be a point on XY but not in S. Let PZ meet QY at T. By definition, the segment QY and thus T is in S. As S is visible from P, the segment PT and so Z is in S. The contradiction shows that S is visible from all points of PQ. Applying this to the triangle ABC, one sees that S is visible from all points of the sides of the triangle, and since every point D of the triangle is on a segment whose endpoints are on the sides, S is visible from D.

N1. Since f is non-deceasing, $f(7)f(13) = f(91) \geq f(90) = f(9)f(10)$ and $f(8)f(9) = f(72) > f(70) = f(7)f(10)$. Multiply the inequalities and cancel by $f(7)f(9) > 0$ to get $f(8)f(13) \geq f(10)f(10) = (f(10))^2$. [In fact, by a theorem of Erdös, in *On the distribution function of additive functions*, Ann. of Math. (2) 47 (1946), 1–20, every multiplicative non-decreasing $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ is of the form $f(n) = n^k$ for some $k \in \mathbb{N}$.]

N2. Let $\angle AOY = \alpha$, $\angle AOZ = \beta$ and $\angle ZOB = \gamma$. Then $\alpha + \beta + \gamma = 180°$. Also $\angle BOX = \alpha$ (vertical angles) and $\angle ACY = \alpha = \angle BCX$ (angles subtending equal arcs) and analogously $\angle COX = \beta$, $\angle ABZ = \beta = \angle CBX$; $\angle COY = \gamma$, $\angle BAZ = \gamma = \angle CAY$. Each of the triangles CYA, CBX and ZBA have two angles of from the set $\{\alpha, \beta, \gamma\}$. The triangles are similar. The similarity implies



$$\frac{AY}{CY} = \frac{AB}{BZ}, \quad \frac{CX}{BX} = \frac{AZ}{AB}.$$

So

$$\frac{AY}{AZ} \cdot \frac{BZ}{BX} \cdot \frac{CX}{CY} = \frac{AB}{BZ} \cdot \frac{AZ}{AB} \cdot \frac{BZ}{AZ} = 1.$$

N3. a) Richard may choose an arbitrary pair A, B of lamps, turn them on, and then use 2^{2008} different combinations of the remaining lamps; then he can turn the two lamps off and again use 2^{2008} different combinations of the other lamps. Thus after 2^{2009} rounds Laura cannot know which button corresponds to A and which to B. But in $2^{2009}+1$ rounds, there has to be at least one combination in which A is turned on while B is off or vice versa. Thus, after $2^{2009}+1$ rounds Laura knows the correspondence of the lamps and buttons.

b) Since $2^{11} = 2048 > 2010$, the lamps and the buttons can be labeled with numbers with 11 binary digits. Pushing in turn buttons with k:th binary digit 1, Laura can find out the label of any lamp. So 11 rounds are enough.

N4. If n has a representation by m simple numbers, then it has a representation by $m+1$ simple numbers: if $n = a_1 \pm a_2 \pm \ldots \pm a_m$ and 10^r is larger than any of the a_j, then, setting $a'_m = a_m + 10^r$, $n = a_1 \pm \ldots \pm a'_m - 10^r$ is a representation of n by $m+1$ simple numbers. Now let n be an arbitrary positive integer. Let a_j be the number with 1 in those decimal places where n has a decimal $\geq j$ and 0 in other places. Evidently $n = a_1 + a_2 + \ldots a_9$, where every a_j is either simple or zero. By what was said above, n has a representation by 9 simple numbers.

To show that there are numbers requiring 9 simple numbers for their representation, consider $n = 10203040506070809$. Assume $n = a_1 + a_2 + \cdots + a_j - a_{j+1} - a_{j+2} - \cdots - a_k$ where each a_i is simple and $k < 9$. Then all digits of $b_1 = a_1 + \cdots a_j$ are $\leq j$ and all digits of $b_2 = a_{j+1} + \ldots + b_k$ are $\leq k - j$. We have $b_1 = n + b_2$. Perform the column addition of n and b_2 and consider the digit $j+1$ of n in the addition. There will be no carries from the lower digits, since the sum there is less than $10\ldots0 + 88\ldots8$ or at most $98\ldots88$. So in the column sum we just add $j+1$ and get a digit of b_1 which is at most j. The contradiction shows that $k \geq 9$.