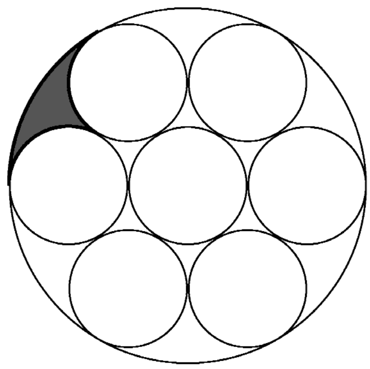
High School Mathematics Contest, First Round

Open Division, Nov. 1, 2011

1. In the figure, the radius of the big circle is 6, the small circles are of equal size, and the outermost and innermost circles are tangent to the other circles. Determine the area of the shaded part of the figure.



2. Solve the Diophantine equation

$$x^2 + (10y - y^2)^2 + y^6 = 2011,$$

i.e. find the integer solutions of the equation.

3. Let $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \frac{x^2 - 2011x + 1}{x^2 + 1}.$$

Show that $|f(x) - f(y)| \leq 2011$ for all real numbers x and y.

4. The plane is tiled with black and white unit squares in such a way that adjoining tiles either have a side or a vertex in common. A line segement in the plane is white, if there exist white squres such that the segment is completely inside them, if the points of intersection of the segment and the sides of the squares are neglected. A black line segment is defined in an analogous manner. Show that the plane can be tiled in such a way that no white or black segment is longer than 5.

High School Mathematics Contest, First Round

Intermediate Division, Nov. 1, 2011

1. Three gamblers playd on money. At the start of the game they had money in ratios $6:5:4$ and at the end they money in ratios $7:6:5$. One of the gamblers won 3 euros. How many euros did he have at the end?

a) 72 b)75 c) 90 d) 108

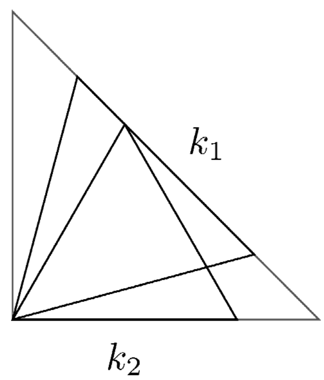
2. What can be inferred of the solutions of the equation $x^{2011}+x+1=0$?

a) The equation has a unique real solution.
b) The equation has at least one rational solution.
c) The equation has no negative solutions.
d) All solutions of the equation lie in the interval $[-1, 1]$.

3. Which of the following claims concerning the number 211 are true?

a) 211 is a prime number. b) 211 is the product of two prime numbers.
c) 211 is the sum of two prime numbers. d) 211 is the sum of three prime numbers.

4. The legs of an isosceles right triangle are of length a. Inside this triangle there are two equilateral trinaglses k_1 and k_2. One vertex of both triangles coincides with the vertex of the right angle. One side of k_1 lies on the hypotenuse and one side of k_2 lies on a leg and one vertex of k_2 is on the hypoptenuse. Determine the ratio of the lengths of the sides of k_1 and k_2.



5. Solve the equation

$$(x^2+y^2-8)^2(1-xy)^2+\sqrt{x^2-y^2}=0.$$

6. Does there exist a positive integer n such that the factorial $n!$ has exactly 154 zeroes at the end? (The *factorial* $n!$ is the product $1\cdot 2\cdot 3\cdots(n-1)\cdot n$.)

Intermediate Division, Answer Sheet

The answers of the multiple choice problems (problems 1 to 3) should be written on this sheet; the answers to the traditional type problems (4 to 6) should be written on a separate sheet. The number of correct alternatives in each multiple choice problem can be anything from 0 to 4. Mark the appropriate square with a +, if you think the alternative is right and a −, if you think the alternative is wrong. Each correct marking is awarded by one point, an incorrect or ambiguous marking yields zero points. The maximum score for each of the problems 4 to 6 is six points.

Working time is 120 minutes. Write your name and the name of your school also on the sheets on which you answer problems 4 to 6.

Name: ______________________________

School: ______________________________

Home address: ______________________________

Email address: ______________________________

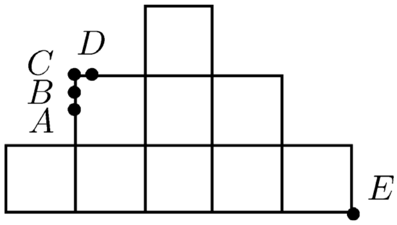
	a	b	c	d
1.				
2.				
3.				

High School Mathematics Contest, First Round

Basic Division, Nov. 1, 2011

1. When the number $5^{140} \cdot 8^{47}$ is written in the usual manner, the number of digits is

a) an odd number b) 47 c) 48 d) 141



2. The adjacent figure is composed of nine squares, each of side 1. The line segments AB, BC and CD have length 1/4. One of the lines AE, BE, CE, DE divides the figure into two parts of equal area. Which one?

a) AE b) BE c) CE d) DE

3. Which of the following statements about the diagonals of a hexagon are true?

a) A hexagon has less than ten diagonals.
b) The diagonals of a convex hexagon can have one point in common.
c) A hexagon can have two non-intersecting diagonals.
d) A regular hexagon has two parallel diagonals.

4. The lengths of the sides of a triangle are $2a$, a^2+1 and a^2-1, where $a > 1$. The triangle then has the properties:

a) The largest angle can be obtuse. b) The largest angle is a right angle.
c) a^2+1 is the longest side. d) The shortest side depends on the value of a.

5. We know that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax^5 + bx^3 + cx + 2$ satisfies $f(3) = 5$. Then one can conclude that

a) $f(0) = 2$ b) $f(-3) = -5$ c) $f(-3) = -1$, d) $f(3) + f(-3) = 8$

6. The digits of the fifth power of an integer can be all different from each other as in $2^5 = 32$ and $3^5 = 243$, or some digits may repeat as in $10^5 = 100000$. The number of positive integers whose fifth powers have all their digits different from each other is

a) at least 70 b) at least 90 c) at most 100, d) more than 1000

7. Two equally long trains on adjoining tracks with velocities u and v, $u > v > 0$. If the trains travel in the same direction their passing time is twice as long as their passing time, when they travel in opposite directions. (The *passing time* is the time when the trains are at least in part beside each other.) Find the ratio u/v.

8. Prove that the inequality $x^6 - x^3 + x^2 - x + 1 > 0$ holds for all real numbers x.

Basic Division, Answer Sheet

The answers of the multiple choice problems (problems 1 to 6) should be written on this sheet; the answers to the traditional type problems (7 and 8) should be written on a separate sheet. The number of correct alternatives in each multiple choice problem can be anything from 0 to 4. Mark the appropriate square with a +, if you think the alternative is right and a −, if you think the alternative is wrong. Each correct marking is awarded by one point, an incorrect or ambiguous marking yields zero points. The maximum score for each of the problems 7 to 8 is six points.

Working time is 120 minutes. Write your name and the name of your school also on the sheets on which you answer problems 7 and 8.

Name: ______________________________

School: ______________________________

Home address: ______________________________

Email address: ______________________________

	a	b	c	d
1.				
2.				
3.				
4.				
5.				
6.				