10^{th} Iranian Geometry Olympiad

October 20, 2023

Contest problems with solutions

10^{th} Iranian Geometry Olympiad

Contest problems with solutions.

This booklet is prepared by Elahe Zahiri, Mahdi Shavali, Amirmohammad Derakhshandeh and Amir Mahdi Mohseni.

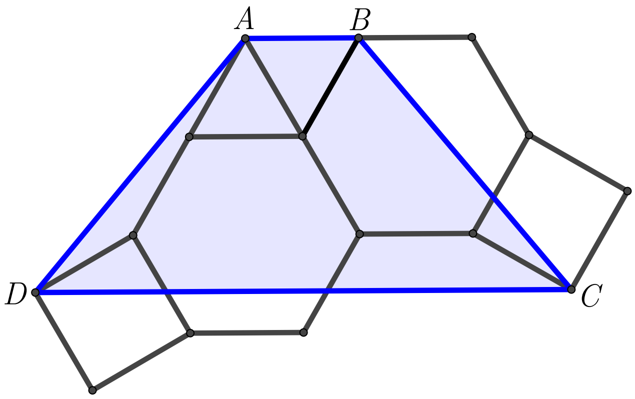
With special thanks to Mahdi Etesamifard and Hesam Rajabzadeh.

Contents

Elementary Level

Problems

Problem 1. All of the polygons in the figure below are regular. Prove that $ABCD$ is an isosceles trapezoid.



(→ p.5)

Problem 2. In an isosceles triangle ABC with $AB = AC$ and $\angle A = 30^\circ$, points L and M lie on the sides AB and AC, respectively such that $AL = CM$. Point K lies on AB such that $\angle AMK = 45^\circ$. If $\angle LMC = 75^\circ$, prove that $KM + ML = BC$.

(→ p.6)

Problem 3. Let $ABCD$ be a square with side length 1. How many points P inside the square (not on its sides) have the property that the square can be cut into 10 triangles of equal area such that all of them have P as a vertex?

(→ p.7)

Problem 4. Let $ABCD$ be a convex quadrilateral. Let E be the intersection of its diagonals. Suppose that $CD = BC = BE$. Prove that $AD + DC \geq AB$.

(→ p.8)

Problem 5. A polygon is decomposed into triangles by drawing some non-intersecting interior diagonals in such a way that for every pair of triangles of the triangulation sharing a common side, the sum of the angles opposite to this common side is greater than 180°.

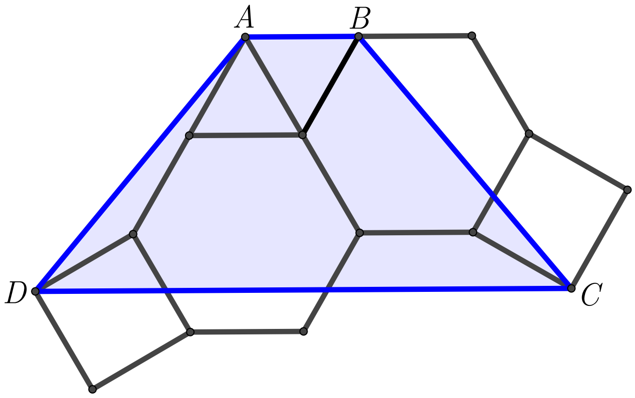
a) Prove that this polygon is convex.

b) Prove that the circumcircle of every triangle used in the decomposition contains the entire polygon.

(→ p.9)

Solutions

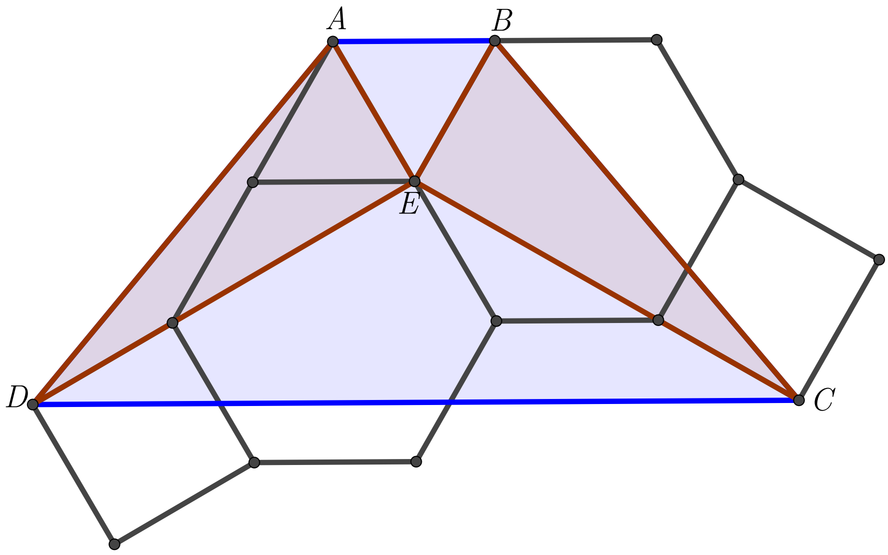
Problem 1. All of the polygons in the figure below are regular. Prove that $ABCD$ is an isosceles trapezoid.



Proposed by Mahdi Etesamifard - Iran

- -

Solution. Notice that the triangles ADE and BEC are congruent. Indeed, $AE = BE$, $DE = CE$ and $\angle AED = \angle BEC = 90°$. Hence $AD = BC$. Notice that $\angle DAE = \angle CBE$ and $\angle EAB = \angle EBA = 60°$. So $\angle DAB = \angle CBA$, implying that $ABCD$ is an isosceles trapezoid.

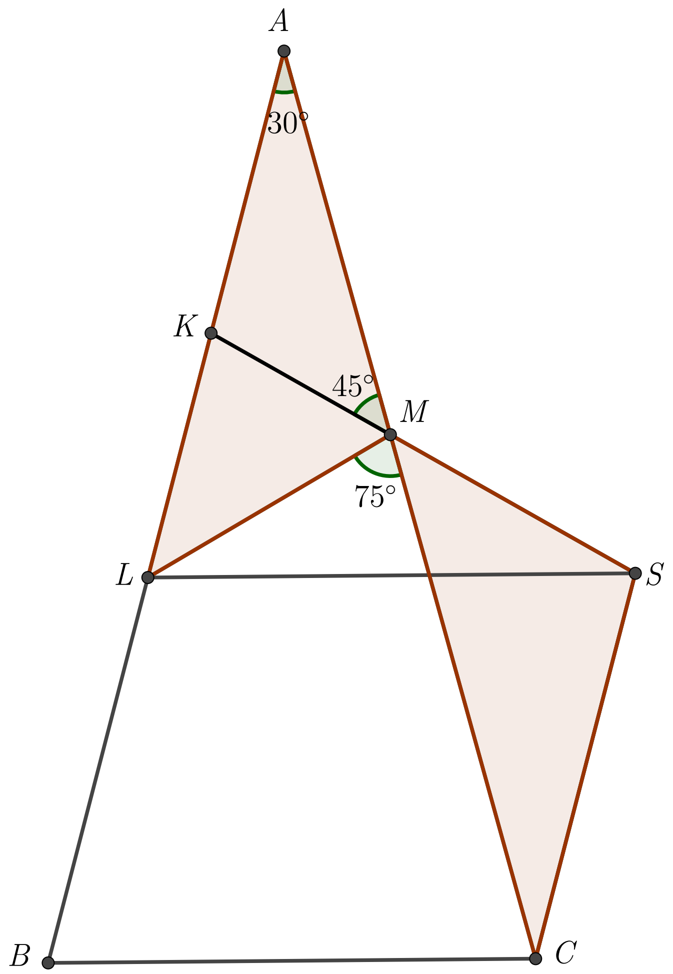


Problem 2. In an isosceles triangle ABC with $AB = AC$ and $\angle A = 30°$, points L and M lie on the sides AB and AC, respectively such that $AL = CM$. Point K lies on AB such that $\angle AMK = 45°$. If $\angle LMC = 75°$, prove that $KM + ML = BC$.

Proposed by Mahdi Etesamifard - Iran

- -

Solution. Let S be the intersection of KM with the line parallel to BC from L. Notice that $\angle MKL = \angle KLS = 75°$. So $\angle MSL = \angle MLS = 30°$, hence $ML = MS$. We have $\angle SMC = \angle AMK = \angle ALM = 45°$ and $MC = AL$. So the triangles ALM and CMS are congruent. So $SC = AM = AC - MC = AB - AL = BL$ and $LS || BC$, hence $BCSL$ is a parallelogram. We had that $\angle MKL = \angle KLS = 75°$ so $KS = SL = BC$, and we are done.



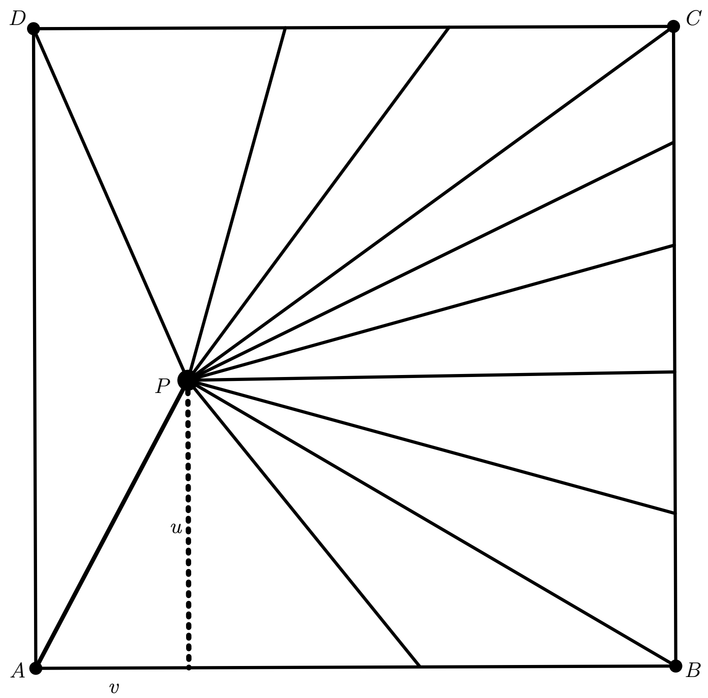
Problem 3. Let $ABCD$ be a square with side length 1. How many points P inside the square (not on its sides) have the property that the square can be cut into 10 triangles of equal area such that all of them have P as a vertex?

Proposed by Josef Tkadlec - Czech Republic

- -

Solution. Answer: 16.

Denote the distances from P to the sides DA and AB by v and u, respectively. Clearly, for each triangle, the side opposite to vertex P will be a part of one side of $ABCD$. Let a, b, c, d be the number of triangles for which the side opposite to P is a part of side AB, BC, CD, DA, respectively. Then, each of the a triangles has area $\frac{1}{2}.\frac{1}{a}.u = \frac{u}{2a}$. Similarly, each of the b triangles has area $\frac{(1-v)}{2b}$, each of the c triangles has area $\frac{(1-u)}{2c}$, and each of the d triangles has area $\frac{v}{2d}$.



Denote the area of a polygon X by $[X]$.

Note that $a + c = b + d$, since the total area of triangles $[ABP]$ and $[PCD]$ satisfies

$$[ABP] + [PCD] = \frac{u}{2} + \frac{1-u}{2} = \frac{1}{2} = \frac{1}{2}[ABCD].$$

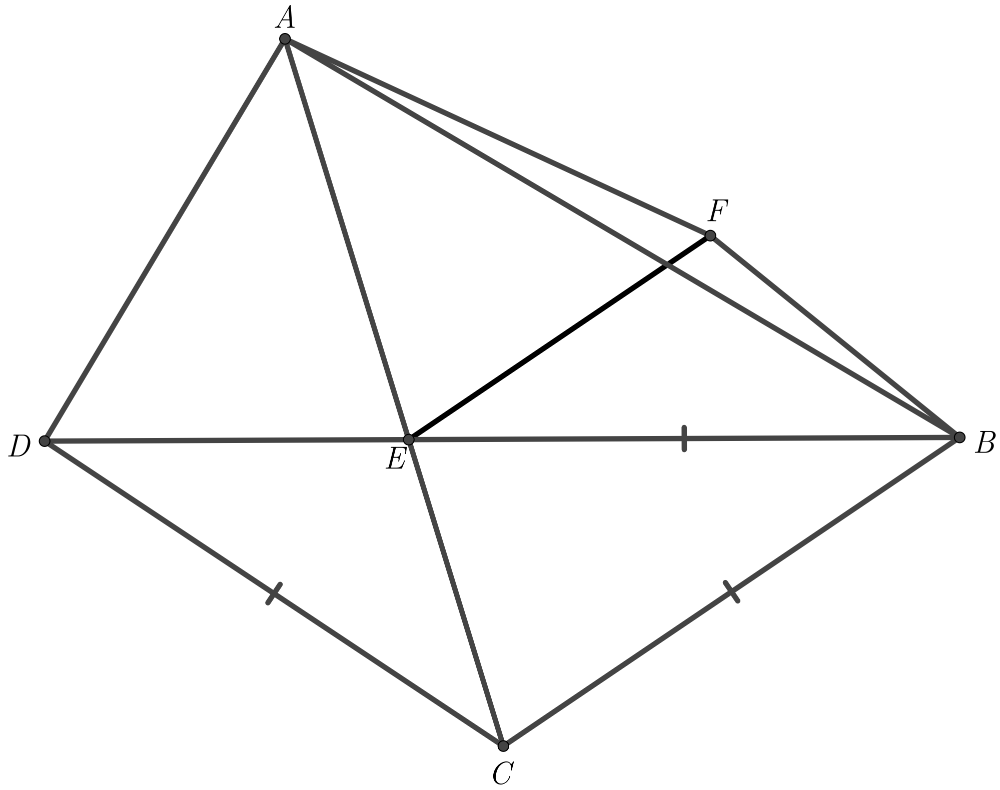
Thus, $a+c = b+d = \frac{1}{2} \times 10 = 5$. On the other hand, for any two pairs (a, c) and (b, d) of positive integers, each with a sum equal to 5, there exists a corresponding point P with distances $u = a/5$ and $v = d/5$ from the sides DA and AB. Since there are $5 - 1 = 4$ pairs of positive integers with a sum equal to 5, the answer is $4 \times 4 = 16$.

Problem 4. Let $ABCD$ be a convex quadrilateral. Let E be the intersection of its diagonals. Suppose that $CD = BC = BE$. Prove that $AD + DC \geq AB$.

Proposed by Dominik Burek - Poland

- -

Solution. Let F be the reflection of D about AC. Notice that $\angle FCB = \angle ECB - \angle ECF = \angle CEB - \angle ECD = \angle CDB = \angle CBE$, $BE = CD = CF$ and $BC = BC$. So the triangles BCE and CBF are congruent. So $BF = CE \leq DC$ and so $AD + DC \geq AF + FB \geq AB$ and



Problem 5. A polygon is decomposed into triangles by drawing some non-intersecting interior diagonals in such a way that for every pair of triangles of the triangulation sharing a common side, the sum of the angles opposite to this common side is greater than 180°.

a) Prove that this polygon is convex.

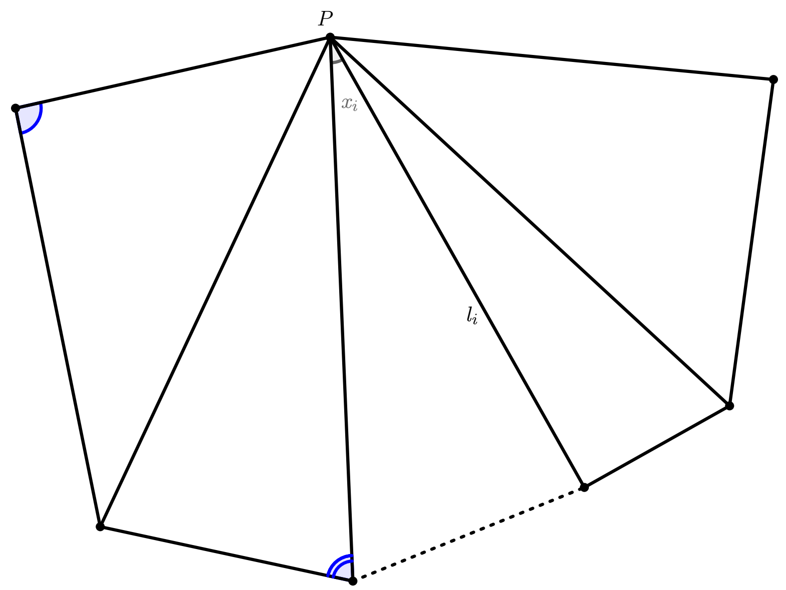
b) Prove that the circumcircle of every triangle used in the decomposition contains the entire polygon.

Proposed by Morteza Saghafian - Iran

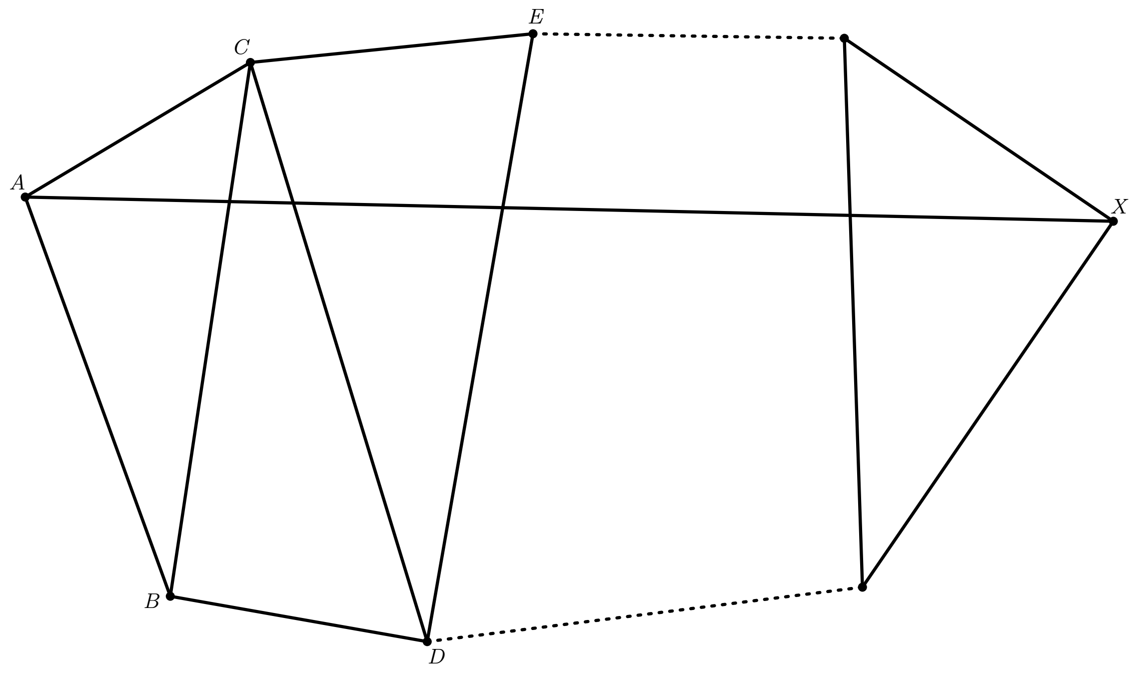
Solution. For an arbitrary vertex P, let $T_1, T_2, ..., T_k$ be all the triangles with P as a vertex from left to right in this order, let $l_1, l_2, ..., l_{k-1}$ be all the diagonals passing through P such that for every $1 \leq i \leq k-1$, $l_i = T_i \cap T_{i+1}$. Finally, denote by x_i the measure of the angle in triangle T_i with P as its vertex.
Note that for every $1 \leq i \leq k-1$ there exists a pair of points B_i, C_i opposite to l_i with angle lengths (b_i, c_i) such that $b_i + c_i > 180$. Now it is easy to see that

$$180k > \sum_{i=1}^{k-1}(b_i + c_i) + \sum_{i=1}^{k} x_i > (k-1)180 + \sum_{i=1}^{k} x_i$$

Therefore, $180 > \sum_1^k x_i$, which means that in this polygon, the angle in P as its vertex is less than 180. Since P was arbitrary, this concludes the first part and shows that the polygon is convex.



Now let ABC be one of the triangles in this decomposition, and let X be a vertex of the polygon outside the circumcircle of ABC. Since this polygon is convex, without loss of generality, suppose AX intersects BC.



Let $\triangle ABC = T_1, T_2, ..., T_k$ be all the triangles that AX intersects, respectively. Denoting $T_2 = \triangle DBC$, we have

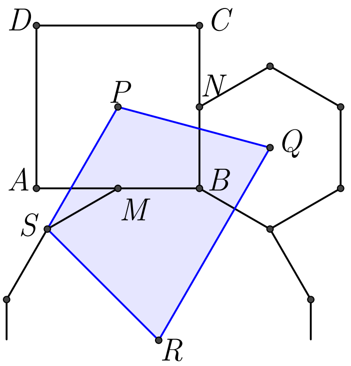
$$\angle BAC + \angle BDC > 180^\circ > \angle BAC + \angle BXC,$$

hence $\angle BDC > \angle BXC$ and X are outside the circumcircle of $\triangle BDC$. With the same approach, we can prove that for every $1 \leq i \leq k$, the point X is outside the circumcircle of triangle T_i, inductively. But, since $\triangle UVW = T_k$ is the last triangle that AX intersects, if X is outside the circumcircle of this triangle, then we would have XVW one of the triangles in the decomposition and $\angle VUW + \angle VXW < 180^\circ$ which is a contradiction. Therefore, X is inside the circumcircle ABC, done.

Intermediate Level

Problems

Problem 1. Points M and N are the midpoints of sides AB and BC of the square $ABCD$. According to the figure, we have drawn a regular hexagon and a regular 12-gon. The points P, Q and R are the centers of these three polygons. Prove that $PQRS$ is a cyclic quadrilateral.



(→ p.15)

Problem 2. A convex hexagon $ABCDEF$ with an interior point P is given. Assume that $BCEF$ is a square and both ABP and PCD are right isosceles triangles with right angles at B and C, respectively. Lines AF and DE intersect at G. Prove that GP is perpendicular to BC.

(→ p.16)

Problem 3. Let ω be the circumcircle of the triangle ABC with $\angle B = 3\angle C$. The internal angle bisector of $\angle A$, intersects ω and BC at M and D, respectively. Point E lies on the extension of the line MC from M such that ME is equal to the radius of ω. Prove that circumcircles of triangles ACE and BDM are tangent.

(→ p.17)

Problem 4. Let ABC be a triangle and P be the midpoint of arc BAC of circumcircle of triangle ABC with orthocenter H. Let Q, S be points such that $HAPQ$ and $SACQ$ are parallelograms. Let T be the midpoint of AQ, and R be the intersection point of the lines SQ and PB. Prove that AB, SH and TR are concurrent.

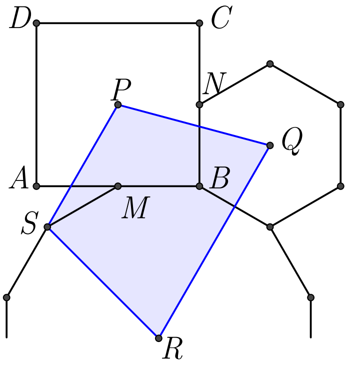
(→ p.18)

Problem 5. There are n points in the plane such that at least 99% of quadrilaterals with vertices from these points are convex. Can we find a convex polygon in the plane having at least 90% of the points as vertices?

(→ p.19)

Solutions

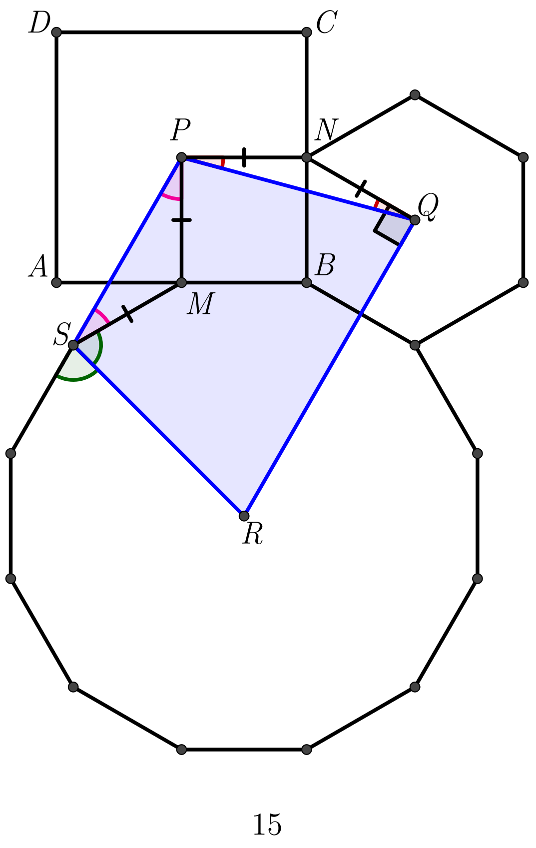
Problem 1. Points M and N are the midpoints of sides AB and BC of the square $ABCD$. According to the figure, we have drawn a regular hexagon and a regular 12-gon. The points P, Q and R are the centers of these three polygons. Prove that $PQRS$ is a cyclic quadrilateral.



Proposed by Mahdi Etesamifard - Iran

- -

Solution. It's easy to see that $\angle NQR = 90°$. Because QR is the perpendicular bisector of BE and $NQ||BE$. Notice that $NP = NQ$ and $\angle PNQ = \angle PNB + \angle BNQ = 90° + 60° = 150°$. So $\angle NQP = 15°$, hence $\angle PQR = \angle NQR - \angle NQP = 90° - 15° = 75°$. Now note that $\angle PMS = 90° + 30° = 120°$ and $PM = MS$, so $\angle PSM = 30°$ and $\angle MSR = 75°$. So $\angle PSR = 105°$, hence $\angle PQR + \angle PSR = 180°$, hence $PQRS$ is cyclic

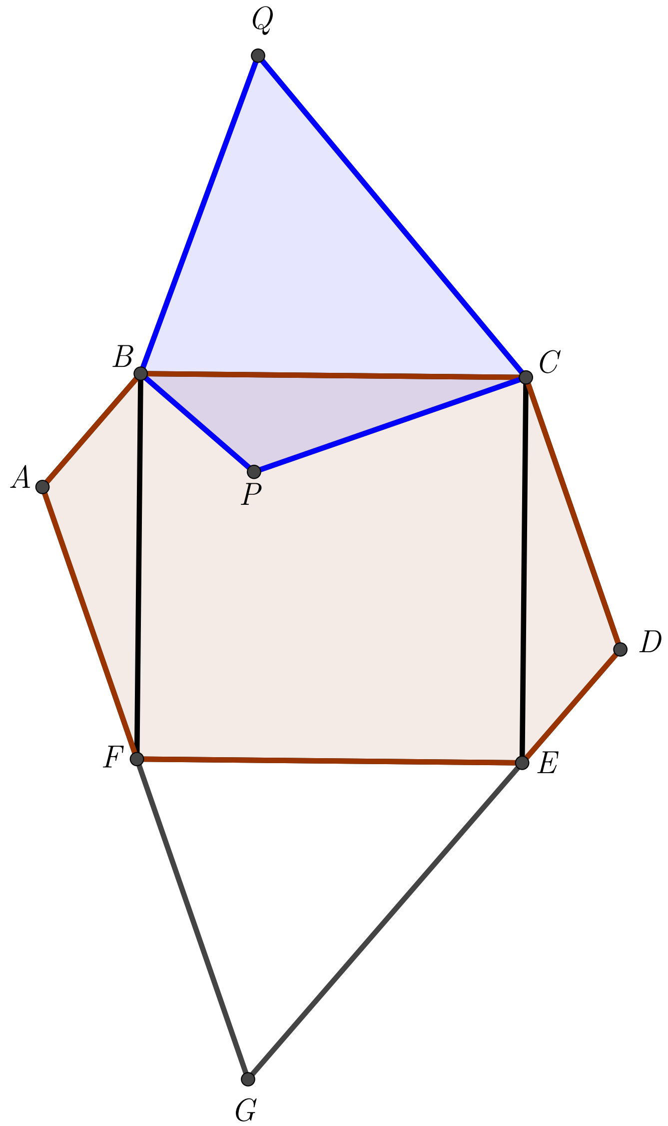


Problem 2. A convex hexagon $ABCDEF$ with an interior point P is given. Assume that $BCEF$ is a square and both ABP and PCD are right isosceles triangles with right angles at B and C, respectively. Lines AF and DE intersect at G. Prove that GP is perpendicular to BC.

Proposed by Patrik Bak - Slovakia

- -

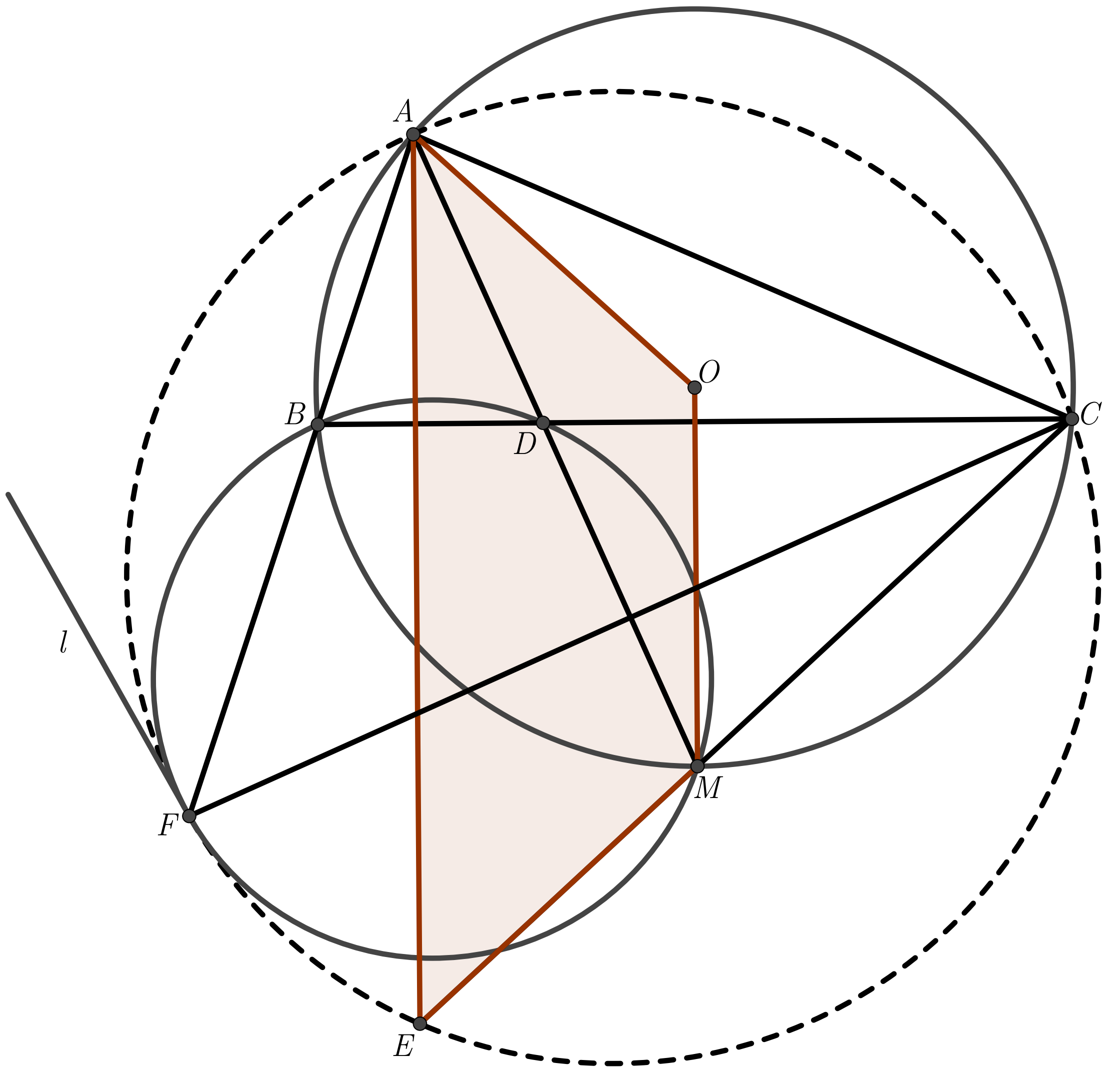
Solution. It is easy to notice that the triangles ABF, PBC, and DEC are congruent. So $\angle BPC = 180° - \angle PBC - \angle PCB = 180° - \angle DEC - \angle AFB = \angle EFG + \angle FEG = 180° - \angle EGF$. Let Q be a point on the other side of BC from G such that the triangles BQC and FGE are congruent. Obviously, $GQ \perp BC$, hence we need to prove $QP \perp BC$. Notice that by the angle chasing we did earlier, the quadrilateral $BPCQ$ is cyclic, and so $\angle BQP = \angle BCP$ and $\angle QBC = \angle EFG = 90° - \angle BFA = 90° - \angle BCP$, hence $\angle BQP + \angle QBC = 90°$ and we are done.



Problem 3. Let ω be the circumcircle of the triangle ABC with $\angle B = 3\angle C$. The internal angle bisector of $\angle A$, intersects ω and BC at M and D, respectively. Point E lies on the extension of the line MC from M such that ME is equal to the radius of ω. Prove that circumcircles of triangles ACE and BDM are tangent.

Proposed by Mehran Talaei - Iran

Solution. Let F be the intersection of AB with the circumcircle of the triangle BDM. First of all, we will prove that $ACEF$ is cyclic. Let $\angle ACB = \alpha$. Notice that $AB.AF = AD.AM = AB.AC$, hence $AF = AC$. So $\angle FCE = 90° - \angle CMA = 90° - 3\alpha$. Note that $AO = OM = ME$ and $\angle AOM = 180° - 2\alpha = 180° - \angle OMC = \angle OME$. So the quadrilateral $AOME$ is isosceles trapezoid. Hence $AE \perp BC$ and $\angle FAE = 90° - 3\alpha$, so $\angle FAE = \angle FCE$ and $ACEF$ is cyclic.
Let l be the line tangent to the circumcircle of $ACEF$. Now note that $\angle BFl = \angle AFl = \angle ACF = 2\alpha$ $\angle BDF = 180° - 2\angle CDM = 180° - 2(90° - \alpha) = 2\alpha$. So $\angle BFl = \angle BDF$, and so l is tangent to the circumcircle of BDM.



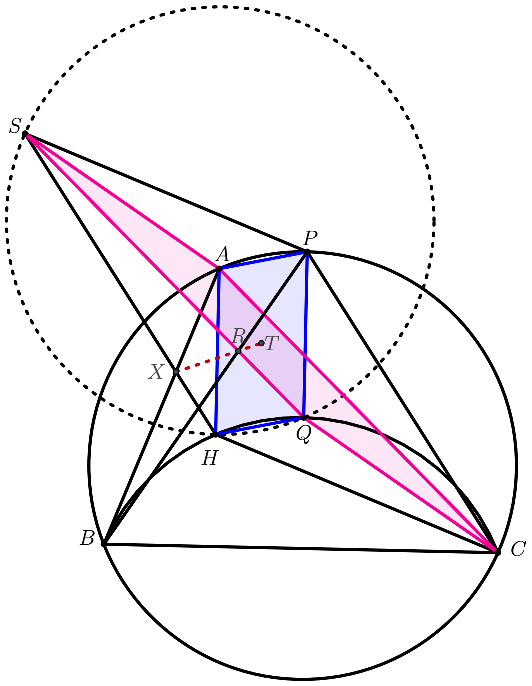
Problem 4. Let ABC be a triangle and P be the midpoint of arc BAC of circumcircle of triangle ABC with orthocenter H. Let Q, S be points such that $HAPQ$ and $SACQ$ are parallelograms. Let T be the midpoint of AQ, and R be the intersection point of the lines SQ and PB. Prove that AB, SH and TR are concurrent.

Proposed by Dominik Burek - Poland

Solution. First, note that the circumcircle of triangle BHC is the translation of the circumcircle of triangle ABC with respect to the vector $\overrightarrow{AH}$. Since $\overrightarrow{AH} = \overrightarrow{PQ}$, $BHQC$ is cyclic.

$$\angle QBC = \angle QCB = \frac{\angle A}{2}, \angle PBQ = \angle PCQ = 90 - \angle A \tag{1}$$

since $AS||QC, SH||PC$ and by (1)



$$\angle ASH = \angle PCQ = 90 - \angle A = \angle ABH$$

hence the quadrilateral $SAHB$ is cyclic and $XA.XB = XH.XS$
same as above since $PS||CH, SQ||AC$ and by (1)

$$\angle PSQ = \angle ACH = 90 - \angle A = \angle PBQ$$

hence the quadrilateral $SPQB$ is cyclic and $RQ.RS = RP.RB$
Now, looking at the circumcircles of triangles SHQ and ABC, points R, X lie on the radical axis of these circles, and since these two circles have the same radii and $AT = TQ, PT = TH$, T should also lie on this radical axis, hence these points are collinear.

Alternative Solution.
Let ζ be the equilateral hyperbola passing through $ABCHP$. By the well-known fact, T is a center of ζ and so $Q, S \in \zeta$. The statement follows from the Pascal theorem on hexagon $APSBQH$

Problem 5. There are n points in the plane such that at least 99% of quadrilaterals with vertices from these points are convex. Can we find a convex polygon in the plane having at least 90% of the points as vertices?

Proposed by Morteza Saghafian - Iran

- -

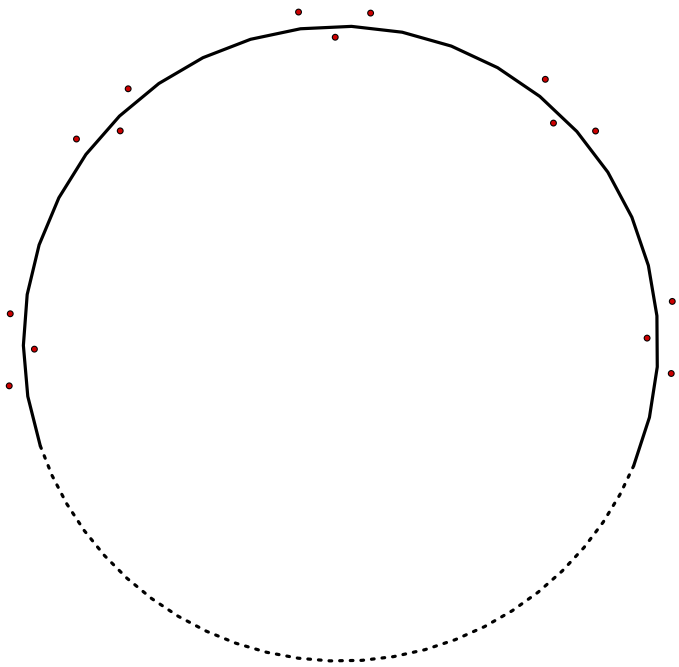
Solution. The answer is no, in general. We provide a counterexample for $n = 3000$ as follows.

Consider a big circle and 1000 triple of points close to vertices of a regular 1000-gon on this inscribed in this circle in such a way that, each triple, shaping the letter "V" faced to the outside of the circle and for every line passing through 2 points in one blob, all the other points are in one side of that line.

Now, with this way of construction, every quadrilateral with vertices from different triples is convex, and so the total number of convex quadruples is at least $3^4\binom{1000}{4}$ On the other hand, the total number of quadruples of points is $\binom{3000}{4}$, and

$$3^4\binom{1000}{4} > \frac{99}{100}\binom{3000}{4}$$

But among these 3000 points, at most 2000 of them can form a convex set. Otherwise, there are 3 points selected from one of the triples, leading to a contradiction.



Advanced Level

Problems

Problem 1. We are given an acute triangle ABC. The angle bisector of $\angle BAC$ cuts BC at P. Points D and E lie on segments AB and AC, respectively, so that $BC \parallel DE$. Points K and L lie on segments PD and PE, respectively, so that points A, D, E, K, L are concyclic. Prove that points B, C, K, L are also concyclic.

(→ p.25)

Problem 2. Let ABC be a triangle with incenter I. The lines BI, CI intersect the sides AC, AB at X, Y, respectively. Let M be the midpoint of the arc BAC of the circumcircle of ABC. Suppose that the quadrilateral $MXIY$ is cyclic. Prove that the area of the quadrilateral $MBIC$ equals the area of the pentagon $BCXIY$.

(→ p.26)

Problem 3. We have chosen a finite number of points, $A_1, A_2, \ldots, A_n$ on the segment S with length L. For each point A_i, let c_i be a closed disk with center A_i and radius less than or equal to 1. Denote the union of c_i's by C. Prove that the perimeter of C is less that $4L + 8$.

(→ p.29)

Problem 4. Let ABC be a triangle with bisectors BE and CF meet at I. Let D be the projection of I on the BC. Let M and N be the orthocenters of triangles AIF and AIE, respectively. Lines EM and FN meet at P. Let X be the midpoint of BC. Let Y be the point lying on the line AD such that $XY \perp IP$. Prove that line AI bisects the segment XY.

(→ p.30)

Problem 5. In triangle ABC points M and N are the midpoints of sides AC and AB, respectively and D is the projection of A into BC. Point O is the circumcenter of ABC and circumcircles of BOC, DMN intersect at points R, T. Lines DT, DR intersect line MN at E and F, respectively. Lines CT, BR intersect at K. A point P lies on KD such that PK is the angle bisector of $\angle BPC$. Prove that the circumcircles of ART and PEF are tangent.

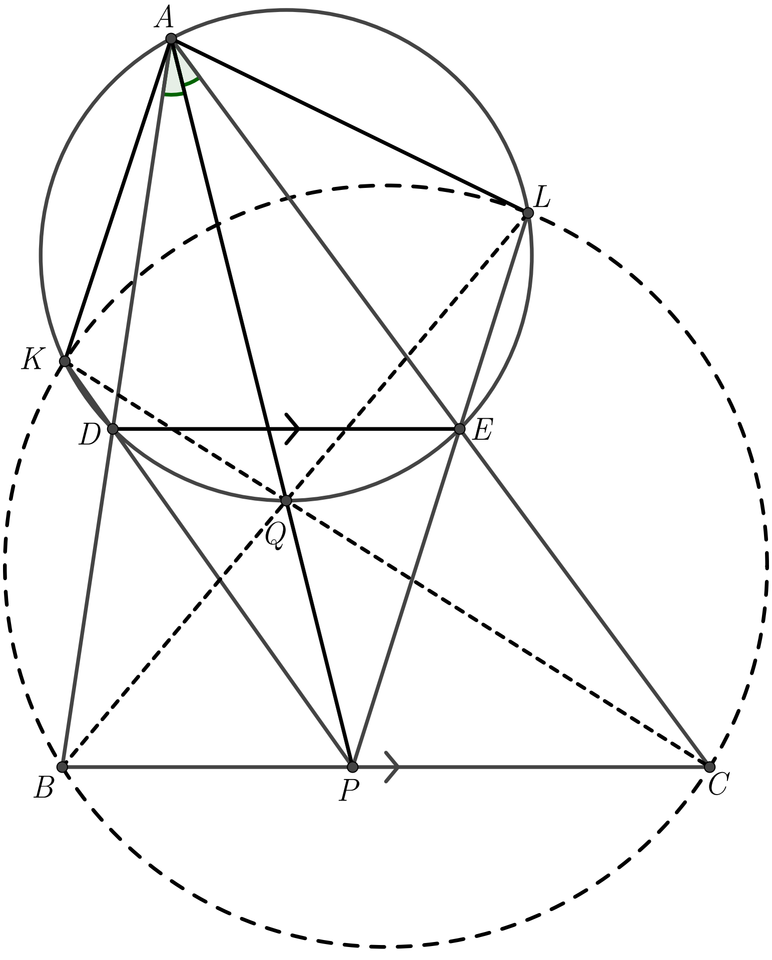
(→ p.33)

Solutions

Problem 1. We are given an acute triangle ABC. The angle bisector of $\angle BAC$ cuts BC at P. Points D and E lie on segments AB and AC, respectively, so that $BC \parallel DE$. Points K and L lie on segments PD and PE, respectively, so that points A, D, E, K, L are concyclic. Prove that points B, C, K, L are also concyclic.

Proposed by Patrik Bak - Slovakia

Solution. Assume that AP intersects the circumcircle of ADE at Q. Notice that $\angle ABP = \angle ADE = 180° - \angle ALP$, so $ALPB$ is cyclic. Hence $\angle ELQ = \angle EAQ = \angle PAB = \angle PLB$. So L, Q, B are collinear. Similarly, $AKPC$ is cyclic, and K, Q, C are collinear. Now note that from $ALPB$ and $AKPC$ being cyclic, we have $AQ.QP = BQ.LQ$ and $AQ.QP = CQ.KQ$ so $BQ.LQ = CQ.KQ$, hence $BCKL$ is cyclic.



Problem 2. Let ABC be a triangle with incenter I. The lines BI, CI intersect the sides AC, AB at X, Y, respectively. Let M be the midpoint of the arc BAC of the circumcircle of ABC. Suppose that the quadrilateral $MXIY$ is cyclic. Prove that the area of the quadrilateral $MBIC$ equals the area of the pentagon $BCXIY$.

Proposed by Dominik Burek - Poland

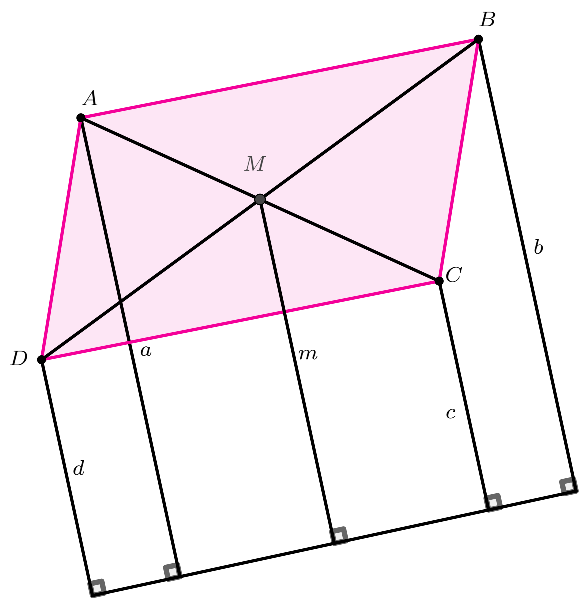
- -

Solution 1. Denote the area of a polygon X by $[X]$. We begin with a lemma:

Lemma. *Let l be an arbitrary line in the plane of the parallelogram $ABCD$. Let a, b, c, d be the distances of A, B, C, D from the line l respectively. Then we have $a + c = b + d$*

Proof. Let M be the midpoint of AC and m be the distance of M to l. then it is easy to see that $a + c = 2m$. Since M is also the midpoint of BD we have $b + d = 2m$ and $a + c = b + d$

□



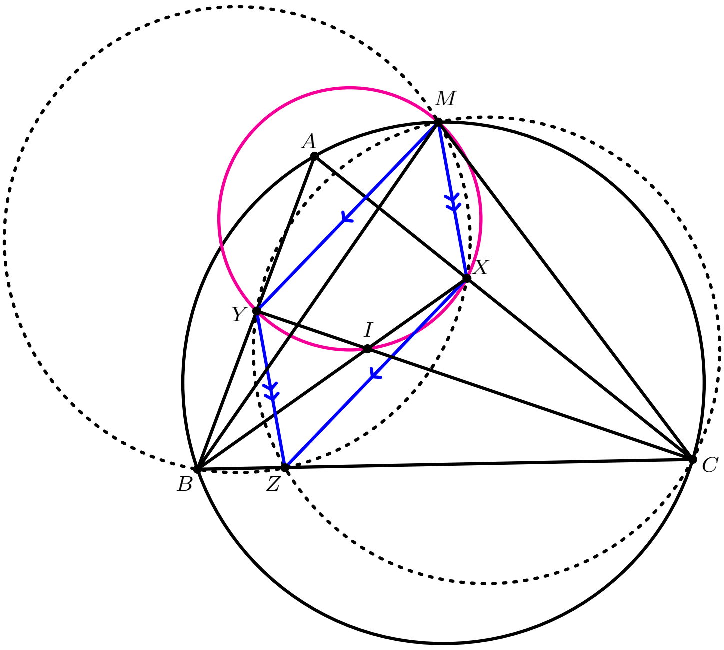
Let the circumcircle of triangle MXB intersect BC at Z.

$$90 - \angle\frac{A}{2} = \angle MBZ = 180 - \angle MXZ$$
$$\angle YMX = 180 - \angle YIX = 90 - \frac{\angle A}{2}$$

Therefore $\angle YMX + \angle MXZ = 180$ hence $MY||XZ$. Since $MXZB$ is cyclic and $MY||XZ$:

$$\frac{\angle C}{2} = \angle MBX = \angle MZX$$
$$\angle XZM = \angle YMZ$$

Therefore the quadrilateral $MYZC$ is cyclic and $YZ||MX$.

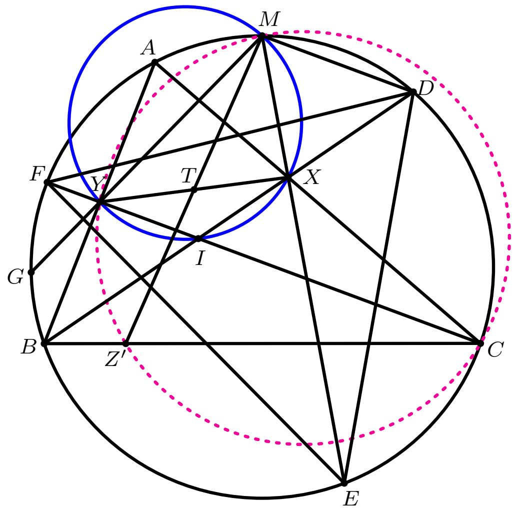


Now $MXZY$ is a parallelogram and by the stated lemma, if x, y, m are the distances of X, Y, M from BC we have $x + y = m$. Now it is easy to see that $[MBC] = [YBC] + [XBC]$. Subtracting $[BIC]$ from both sides of the equality will give us the desired result.

Solution 2.

We denote the area of a polygon F by $[F]$. We must prove that $[MBIC] = [BCXIY]$. Adding to both sides $[BIC]$ we obtain equivalently $[BCM] = [BCX] + [BCY]$. Dividing by $\frac{BC}{2}$ we obtain $dist(X, BC) + dist(Y, BC) = dist(M, BC)$. Let T be the midpoint of XY . Then $dist(X, BC) + dist(Y, BC) = 2dist(T, BC)$, we have to prove that $2dist(T, BC) = dist(M, BC)$, i.e. that the point symmetric to M with respect to T lies on BC. Denote that point by Z. Then $MXZY$ is a parallelogram.

Denote the circumcircle of ABC by Ω. Let BX, MX intersect Ω again at D, E, respectively. Let CY, MY intersect Ω again at F, G, respectively. First we show that $MB = MC = DF = GE$. To this end, we prove that the arcs MB, CM, DF, and GE all subtend the angle $90° - \frac{\angle BAC}{2} = \frac{\angle CBA + \angle ACB}{2}$. This is clear for the arcs MB and MC because $MB = MC$ and $\angle BMC = \angle BAC$. For the arc DF, note that it subtends an angle equal to $\angle DBA + \angle ACF = \frac{\angle CBA + \angle ACB}{2}$. Finally, observe that $\angle GME = \angle YMX = 180° - \angle XIY = 180° - \angle BIC = \angle CBI + \angle ICB = \frac{\angle CBA + \angle ACB}{2}$. Consequently, the arcs DM and FB subtend equal arcs as well.



We have $\angle EMD = \angle EFD$ and $\angle MDX = 90° - \angle BAC = \angle DEF$. It follows that$\triangle XMD \sim$

$\triangle DFE$. Hence
$$\frac{MX}{MD} = \frac{FD}{FE}$$
Similarly, $\angle FMG = \angle FDG$ and $\angle CFM = \angle DGF$. It follows that $\triangle YMF \sim \triangle FDG$ and
$$\frac{MY}{MF} = \frac{DF}{DG}$$
Since $FE = DG$, the above equalities imply that
$$\frac{MX}{MD} = \frac{FD}{FE} = \frac{DF}{DG} = \frac{MY}{MF}$$
Let the circumcircle of MYC intersect BC again at Z'. Since the arcs DM and FB subtend equal angles, we have $\angle DFM = \angle FCB = \angle YCZ' = \angle YMZ'$. Moreover, $\angle MDF = \angle MCF = \angle MCY = \angle MZ'Y$. It follows that $\triangle FDM \sim \triangle MZ'Y$, so
$$YZ' = MD \cdot \frac{MY}{FM} = MD \cdot \frac{MX}{MD} = MX$$
This along with $\angle Z'YM = 180° - \angle MCZ' = 180° - \angle YMX$ implies that $Z'YMX$ is a parallelogram. therefore $Z' = Z$ and we are done.

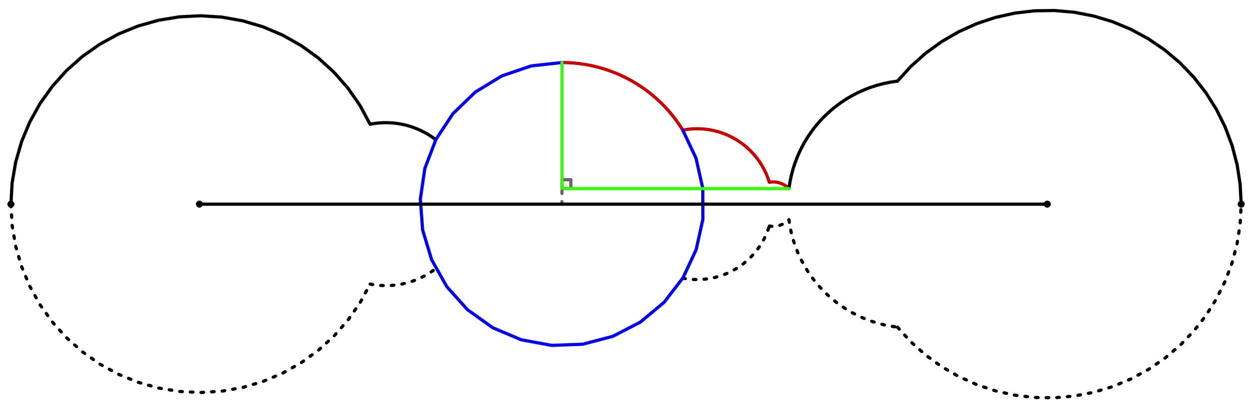
Problem 3. We have chosen a finite number of points, $A_1, A_2, \ldots, A_n$ on the segment S with length L. For each point A_i, let c_i be a closed disk with center A_i and radius less than or equal to 1. Denote the union of c_i's by C. Prove that the perimeter of C is less that $4L + 8$.

Proposed by Morteza Saghafian - Iran

- -

Assume that S is horizontal. Note that the union of the disks is invariant under reflection across this line. Think of the boundary above the line containing S as the graph of a function, with alternating minima and maxima as we go from left to right. We focus on the piece of the graph between a minimum and an adjacent maximum (the red piece in Figure) and claim that this piece is at least as wide as it is high. (Note that this is not necessarily true for all the arcs on the boundary, but it is true for all paths between a minimum and an adjacent maximum.)

To see this, note that the maximum is the center of a disk, and the piece lies on or above the upper half-circle in the boundary of this disk (the blue disk in the figure). If the entire piece lies in this half-circle, and the minimum is where the half-circle touches the line, then the width equals the height. In all other cases, the width exceeds the height because the horizontal projection of the piece to this half-circle is as wide as it is high.
The length of the piece is less than its width plus its height, which is at most twice the width. The sum of widths is at most $L + 2$, which implies that the length of the union of disks above the line containing S is less than $2L + 4$.



We get the same bound for the length below the line containing S, which implies the statement.

Problem 4. Let ABC be a triangle with bisectors BE and CF meet at I. Let D be the projection of I on the BC. Let M and N be the orthocenters of triangles AIF and AIE, respectively. Lines EM and FN meet at P. Let X be the midpoint of BC. Let Y be the point lying on the line AD such that $XY \perp IP$. Prove that line AI bisects the segment XY.

Proposed by Tran Quang Hung - Vietnam

- -

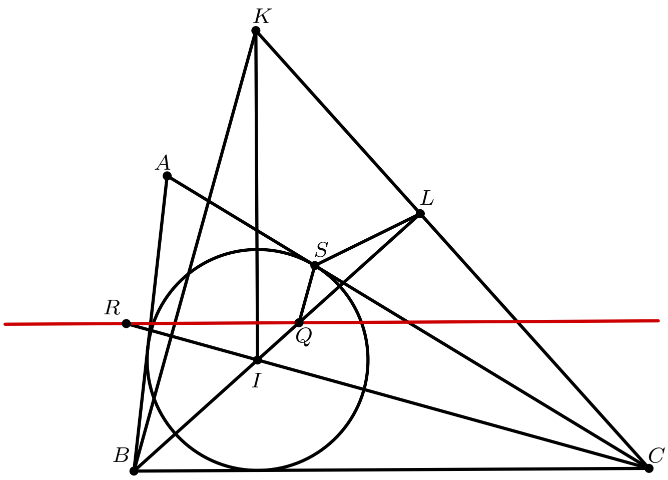
Solution. We need three Lemmas:

Lemma 1. Let ABC be a triangle with incircle (I). K is orthocenter of triangle IBC. Then, the polar of K with respect to circle (I) is the A-midline of triangle ABC.

Proof. Let Q, R, be the feet of perpendicular lines from A to the lines IB, IC, respectively. Easily seen line QR is A-midline of triangle ABC. We shall prove that QR is polar of K with respect to the circle (I). Indeed, circle (I) touches CA at S, triangle AIQ is right at Q, we obtain the quadrilateral $ISLC$ and $IQSA$ are cyclic. We have:

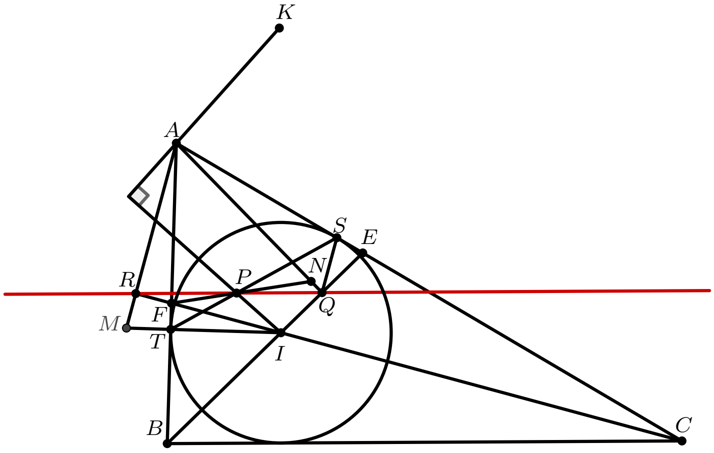
$$\angle ISQ = \angle IAQ = \angle AIB 90^\circ = \angle ICA = \angle ILS$$

From this, $IQ \cdot IL = IS^2$, but CK is perpendicular to IQ at L, this remains that CK is polar of Q with respect to (I). Thus, K and Q are conjugate with respect to (I). Similarly, K and R are conjugate with respect to (I). Therefore QR is polar of K with respect to (I). This completes the proof of Lemma 1. □



Lemma 2. Let ABC be a triangle with bisectors BE, CF meet at I. Let M, N, and K be orthocenters of triangles AIE, AIF, and IBC, respectively. Let ME meet NF at P. Then, lines IP and AK are perpendicular.

Proof. Let incircle (I) touchCA, AB at S, T, respectively. Let the lines AM, AN meet the lines IC, IB at R, Q, respectively. Easily seen S, Q, T, R lie on a circle with diameter AI. Apply Pascal's theorem for six points $\binom{RSI}{TQA}$, and we deduce that lines ME, QR, and ST are concurrent. Similarly, lines NF, QR, and ST are concurrent.
Hence, lines ME, NF, QR, ST are concurrent at P. Notice that A is the pole of EF with respect to (I). It follows from Lemma 1, K is the pole of QR with respect to (I), thus P is the pole of AK with respect to (I) or $IP \perp AK$. This completes the proof of Lemma 2. □



Lemma 3. Let ABC be a triangle. Incircle of ABC touches BC at D. J is excenter at vertex A of ABC. M is the midpoint of ABC. Then, lines JM and AX are parallel.

Proof. Let E be the tangent point on side BC of A-excircle (J) of ABC. Let EF be diameter of (J). Consider the homothety center A such that (I) transform to (J), thus D be tranformed to F so that A, D, F are collinear. Because D and E be tangent point of incircle and A-excircle so:

$$DB = \frac{BC + CAAC}{2} = CE$$

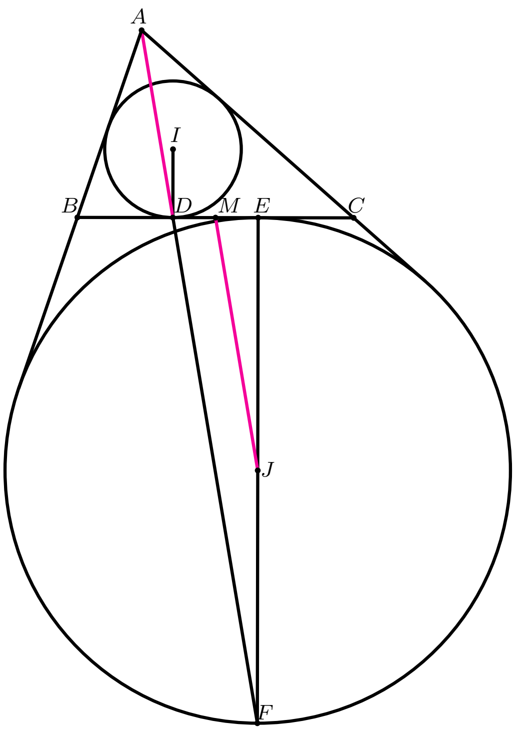
Thus, M is the midpoint of DE. Also J is midpoint of EF. Hence follow midline theorem

$$JM \parallel AF$$

or

$$JM \parallel AD.$$

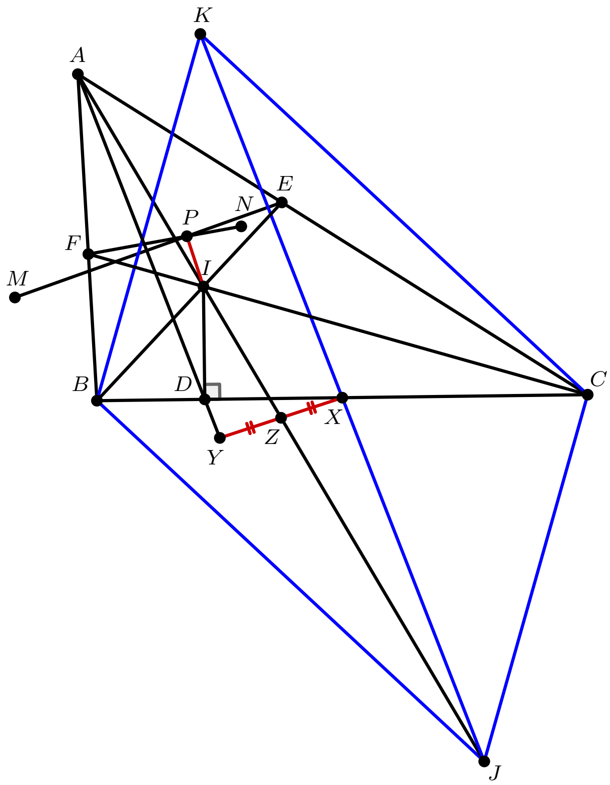
This completes proof of Lemma 3. □



Coming back to the problem.
Let J be the A-excenter of triangle ABC. Let K be the orthocenter of triangle IBC. Easily seen $KBJC$ is a parallelogram, so X is the common midpoint of JK and BC. It follows from Lemma 2, we have $AK \perp IP$. Combining with the assumption $XY \perp IP$, we get $XY \parallel AK$. (1)
It follows from Lemma 3, we have $XJ \parallel AD$ or $XK \parallel AY$. (2)
From (1) and (2), we deduce that $AKXY$ is a parallelogram. Therefore, we have equal vectors

$$\overrightarrow{AY} = \overrightarrow{KX} = \overrightarrow{XJ}$$

From this $AXJY$ is a parallelogram or AI bisects segment XY. This completes the proof of the problem.



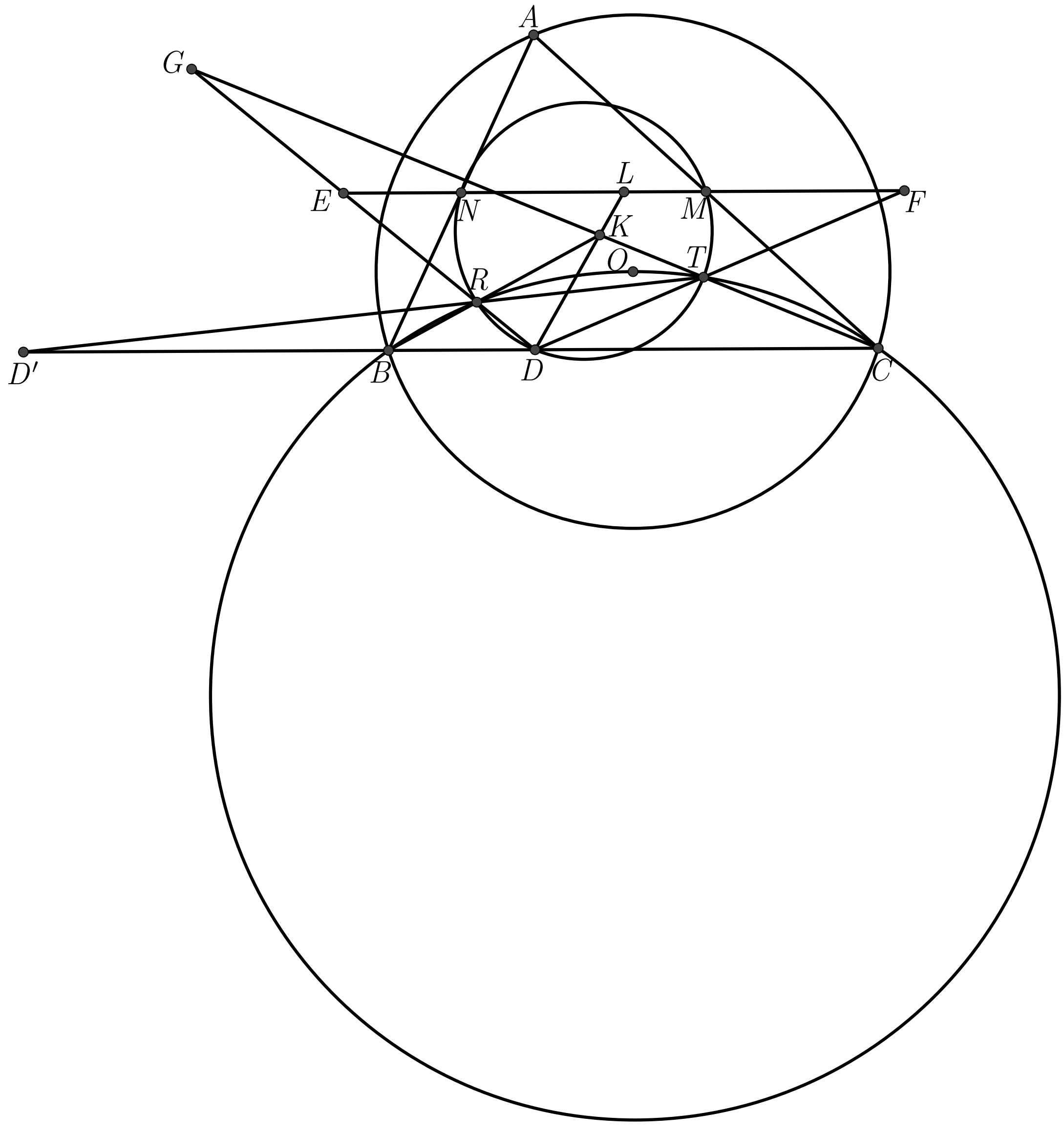
Problem 5. In triangle ABC points M and N are the midpoints of sides AC and AB, respectively and D is the projection of A into BC. Point O is the circumcenter of ABC and circumcircles of BOC, DMN intersect at points R, T. Lines DT, DR intersect line MN at E and F, respectively. Lines CT, BR intersect at K. A point P lies on KD such that PK is the angle bisector of $\angle BPC$. Prove that the circumcircles of ART and PEF are tangent.

Proposed by Mehran Talaei - Iran

- -

Solution.
Claim 1. The line DK bisects EF.

Proof. First of all, let RT intersect BC at D'. From $BRTC$ being cyclic we can conclude that $(BC, DD') = -1$. Now projecting to the line CT from R we see that if RD intesects TC at G, we have $(CK, TG) = -1$. So looking from D we notice that $D(CK, EF) = D(CK, TR) = D(CK, TG) = -1$. Hence from $BC||EF$ we conclude that DK bisects EF.



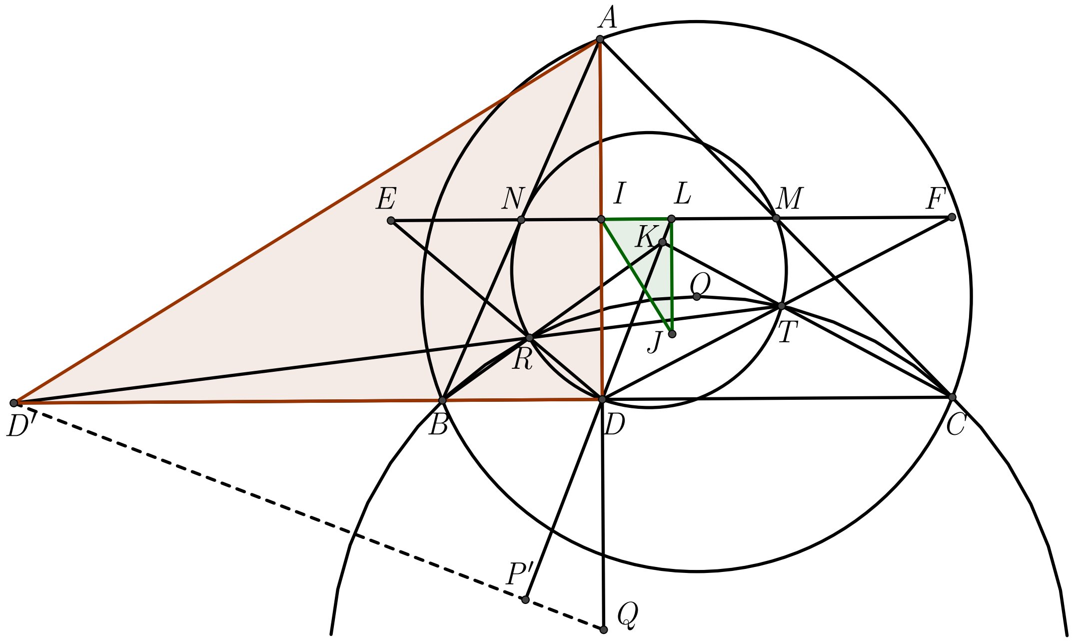
□

Let L be the midpoint of EF. In the course of the proof of the claim, we proved that $D(CK, EF) = -1$. So by projecting to RT and then projecting from K to BC, we see that $(D'D, BC) = -1$. Hence DK is the polar of D' with respect to the circle BOC. Now let P' be a point on the median DK of triangle EF such that $LD.LP = LE.LF$ and Q be the orthocenter of the triangle

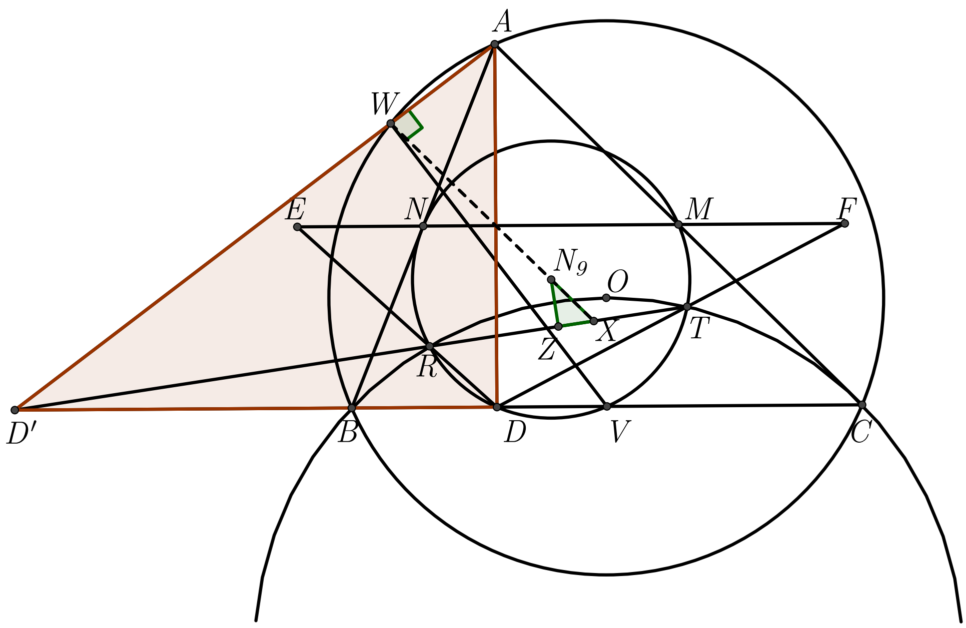
DEF. It's well known that $P'Q \perp DK$.

Claim 2. D', P', Q are collinear.

Proof. Notice that we must prove that the triangles $D'DQ$ and DPQ are similar. But DPQ is similar to DIL where I is the intersection of MN with AD. So, it is enough to prove that $\frac{D'D}{DI} = \frac{DQ}{IL}$. Hence if J is the circumcircle of the triangle AEF then $2JL = DQ$ and $AD = 2ID$ and it is sufficient to prove $\frac{D'D}{AD} = \frac{LJ}{IL}$ or equivalently the triangles ADD' and ILJ are similar.

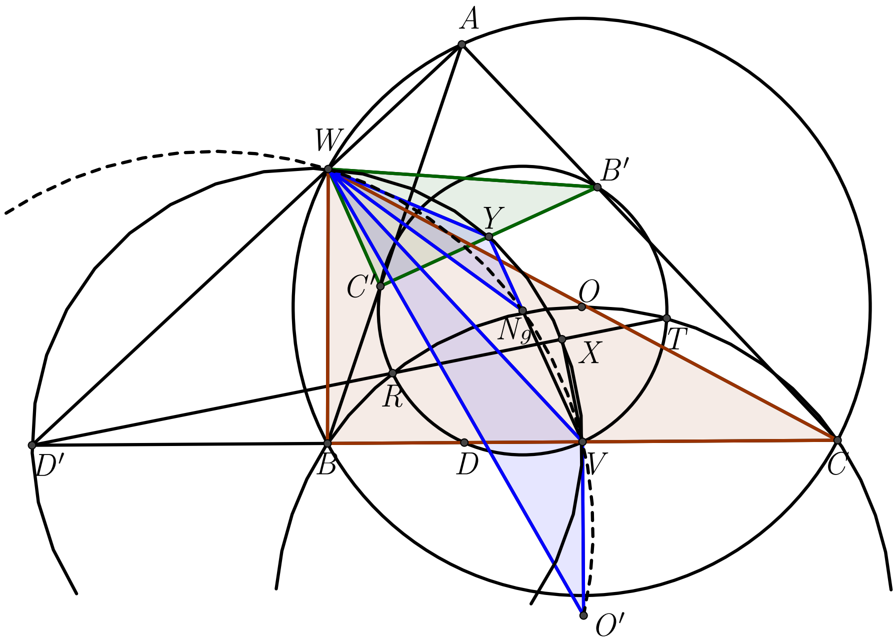


Now Let V be the midpoint of BC. Then $\angle VRT = \angle VDT = \angle DFE$ and similarly $\angle VTR = \angle DEF$, hence the triangles VTR and DEF and AEF are similar. Now notice that the triangles ILJ in AEF is similar to the triangle with analogous points in VTR. So if N_9 is the center of the 9-point circle of ABC, Z is the midpoint of RT and X is the projection of V onto RT, we must prove XN_9Z is similar to ADD'. Now Let W be the intersection of AD' with the circumcircle of ABC. Hence by well known facts $\angle D'AD = \angle WVD'$ and we must prove it is equal to $\angle N_9XD'$ or equivalently W, N_9, X are colinear.



So we must prove $\angle N_9WV = \angle XWV$. Notice that since $\angle SWV = 90^\circ = \angle VXS$ and $SWXV$ is cyclic then $\angle XWV = \angle XSV = \angle VO'Z$ where O' is the circumcircle of BOC. Hence, it is sufficient to prove WN_9VO' is cyclic.

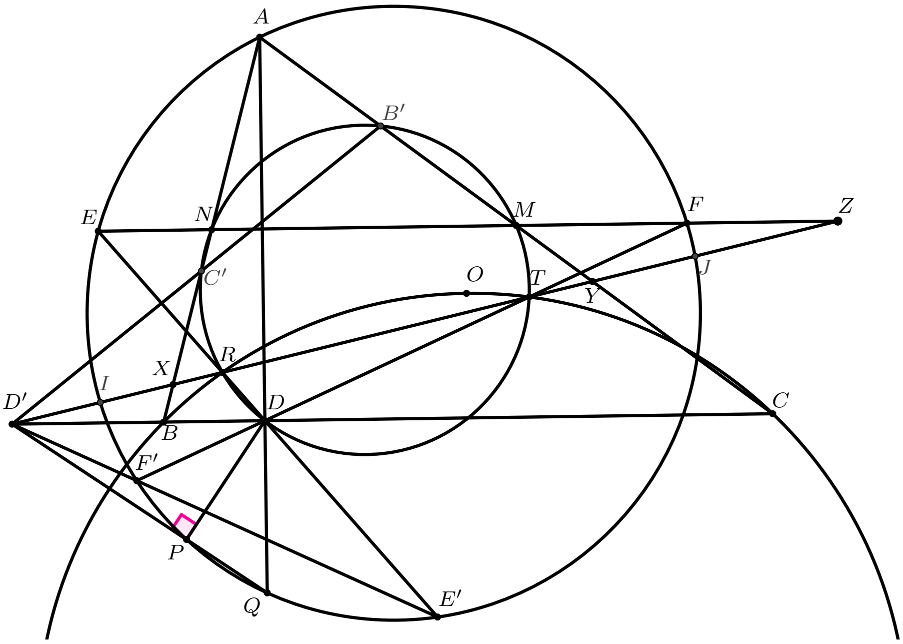
Let C', B' be the projections of C, B to AB, AC respectively, and let Y be the midpoint of $B'C'$. It's easy to see that the $WC'B'$ is similar to WBC and N_9 and O' are analogous points for these triangles. Similarly, since Y, V are analogous points of these two triangles, then WYN_9 and WVO' are collinear, and since $VB' = VC'$ then V, N_9, Y are collinear, then $\angle YN_9W = \angle VO'W$ and so WN_9VO' is cyclic and we are done.



□

Since $P(D'D, BC) = -1$ and by colinearity of D', P', Q we conclude that PD must be the angle bisector of $\angle P'BC$ so $P \equiv P'$ and so $AEPF$ is cyclic.

Let the lines DE, DF intersect with the circumcircle AEF at E', F'. Since $EF||BC$, the circumcircle of triangle $DE'F'$ is tangent to the line BC at D. The circumcircle of triangle DPQ is also tangent to the line BC at D since $\angle QDD' = \angle DPD' = 90$. Therefore D' is the radical center of the three circumcircles $AEF, DPQ, DE'F'$; hence, points D', E', F' are collinear. Let the line RT intersect AB, AC, EF at X, Y, Z respectively, and suppose this line intersects with the circumcircle of AEF at two points I, J.



Now by *Desargues' involution theorem* on the line RT and the quadrilateral $EFE'F'$ there is an involution f swapping $(R,T),(D',Z),(I,J)$. We know that B',C',D' are collinear, hence by *Desargues' involution theorem* on the line RT and the quadrilateral $B'MC'N$ there is an involution g swapping $(R,T),(D',Z),(X,Y)$. Since f,g share the two pairs $(R,T),(D',Z)$ they are the same involution therefore the circumcircle of triangles $AIJ,ART,AXY,AD'Z$ are coaxial. Note that $\angle BAR = \angle CAT$ since the circumcircles BOC,DRT can be mapped to each other with an inversion centered at A, radius $\sqrt{\frac{1}{2}AB.AC}$ and a reflection with respect to the angle bisector of $\angle BAC$. Thus, the circumcircles of triangles ART,AXY are tangent. Hence, the circumcircle $AEFIJ$ should also be tangent to ART.