



# Problems

(with solutions)

60<sup>th</sup> International Mathematical Olympiad  
Bath — UK, 11th–22nd July 2019

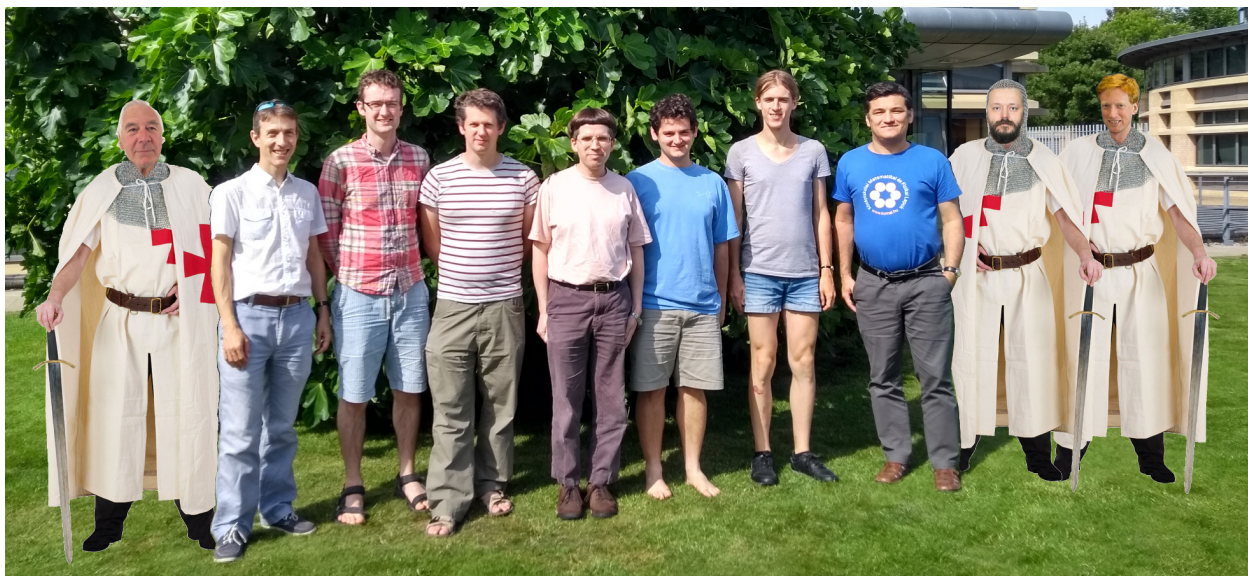


## Contributing Countries

The Organising Committee and the Problem Selection Committee of IMO 2019 thank the following 58 countries for contributing 204 problem proposals:

Albania, Armenia, Australia, Austria, Belarus, Belgium, Brazil, Bulgaria, Canada, China, Croatia, Cuba, Cyprus, Czech Republic, Denmark, Ecuador, El Salvador, Estonia, Finland, France, Georgia, Germany, Greece, Hong Kong, Hungary, India, Indonesia, Iran, Ireland, Israel, Italy, Japan, Kazakhstan, Kosovo, Luxembourg, Mexico, Netherlands, New Zealand, Nicaragua, Nigeria, North Macedonia, Philippines, Poland, Russia, Serbia, Singapore, Slovakia, Slovenia, South Africa, South Korea, Sweden, Switzerland, Taiwan, Tanzania, Thailand, Ukraine, USA, Vietnam.

## Problem Selection Committee



Tony Gardiner, Edward Crane, Alexander Betts, James Cranch, Joseph Myers (chair), James Aaronson, Andrew Carlotti, Géza Kós, Ilya I. Bogdanov, Jack Shotton

# Problems

## Day 1

**Problem 1.** Let  $\mathbb{Z}$  be the set of integers. Determine all functions  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  such that, for all integers  $a$  and  $b$ ,

$$f(2a) + 2f(b) = f(f(a + b)).$$

(South Africa)

**Problem 2.** In triangle  $ABC$ , point  $A_1$  lies on side  $BC$  and point  $B_1$  lies on side  $AC$ . Let  $P$  and  $Q$  be points on segments  $AA_1$  and  $BB_1$ , respectively, such that  $PQ$  is parallel to  $AB$ . Let  $P_1$  be a point on line  $PB_1$ , such that  $B_1$  lies strictly between  $P$  and  $P_1$ , and  $\angle PP_1C = \angle BAC$ . Similarly, let  $Q_1$  be a point on line  $QA_1$ , such that  $A_1$  lies strictly between  $Q$  and  $Q_1$ , and  $\angle CQ_1Q = \angle CBA$ .

Prove that points  $P$ ,  $Q$ ,  $P_1$ , and  $Q_1$  are concyclic.

(Ukraine)

**Problem 3.** A social network has 2019 users, some pairs of whom are friends. Whenever user  $A$  is friends with user  $B$ , user  $B$  is also friends with user  $A$ . Events of the following kind may happen repeatedly, one at a time:

Three users  $A$ ,  $B$ , and  $C$  such that  $A$  is friends with both  $B$  and  $C$ , but  $B$  and  $C$  are not friends, change their friendship statuses such that  $B$  and  $C$  are now friends, but  $A$  is no longer friends with  $B$ , and no longer friends with  $C$ . All other friendship statuses are unchanged.

Initially, 1010 users have 1009 friends each, and 1009 users have 1010 friends each. Prove that there exists a sequence of such events after which each user is friends with at most one other user.

(Croatia)

## Day 2

**Problem 4.**

Find all pairs  $(k, n)$  of positive integers such that

$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}).$$

(El Salvador)

**Problem 5.**

The Bank of Bath issues coins with an  $H$  on one side and a  $T$  on the other. Harry has  $n$  of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly  $k > 0$  coins showing  $H$ , then he turns over the  $k^{\text{th}}$  coin from the left; otherwise, all coins show  $T$  and he stops. For example, if  $n = 3$  the process starting with the configuration  $THT$  would be  $THT \rightarrow HHT \rightarrow HTT \rightarrow TTT$ , which stops after three operations.

- (a) Show that, for each initial configuration, Harry stops after a finite number of operations.
- (b) For each initial configuration  $C$ , let  $L(C)$  be the number of operations before Harry stops. For example,  $L(THT) = 3$  and  $L(TTT) = 0$ . Determine the average value of  $L(C)$  over all  $2^n$  possible initial configurations  $C$ .

(USA)

**Problem 6.**

Let  $I$  be the incentre of acute triangle  $ABC$  with  $AB \neq AC$ . The incircle  $\omega$  of  $ABC$  is tangent to sides  $BC$ ,  $CA$ , and  $AB$  at  $D$ ,  $E$ , and  $F$ , respectively. The line through  $D$  perpendicular to  $EF$  meets  $\omega$  again at  $R$ . Line  $AR$  meets  $\omega$  again at  $P$ . The circumcircles of triangles  $PCE$  and  $PBF$  meet again at  $Q$ .

Prove that lines  $DI$  and  $PQ$  meet on the line through  $A$  perpendicular to  $AI$ .

(India)

# Solutions

## Day 1

### Problem 1.

Let  $\mathbb{Z}$  be the set of integers. Determine all functions  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  such that, for all integers  $a$  and  $b$ ,

$$f(2a) + 2f(b) = f(f(a + b)). \quad (1)$$

(South Africa)

**Answer:** The solutions are  $f(n) = 0$  and  $f(n) = 2n + K$  for any constant  $K \in \mathbb{Z}$ .

**Common remarks.** Most solutions to this problem first prove that  $f$  must be linear, before determining all linear functions satisfying (1).

**Solution 1.** Substituting  $a = 0, b = n + 1$  gives  $f(f(n + 1)) = f(0) + 2f(n + 1)$ . Substituting  $a = 1, b = n$  gives  $f(f(n + 1)) = f(2) + 2f(n)$ .

In particular,  $f(0) + 2f(n + 1) = f(2) + 2f(n)$ , and so  $f(n + 1) - f(n) = \frac{1}{2}(f(2) - f(0))$ . Thus  $f(n + 1) - f(n)$  must be constant. Since  $f$  is defined only on  $\mathbb{Z}$ , this tells us that  $f$  must be a linear function; write  $f(n) = Mn + K$  for arbitrary constants  $M$  and  $K$ , and we need only determine which choices of  $M$  and  $K$  work.

Now, (1) becomes

$$2Ma + K + 2(Mb + K) = M(M(a + b) + K) + K$$

which we may rearrange to form

$$(M - 2)(M(a + b) + K) = 0.$$

Thus, either  $M = 2$ , or  $M(a + b) + K = 0$  for all values of  $a + b$ . In particular, the only possible solutions are  $f(n) = 0$  and  $f(n) = 2n + K$  for any constant  $K \in \mathbb{Z}$ , and these are easily seen to work.

**Solution 2.** Let  $K = f(0)$ .

First, put  $a = 0$  in (1); this gives

$$f(f(b)) = 2f(b) + K \quad (2)$$

for all  $b \in \mathbb{Z}$ .

Now put  $b = 0$  in (1); this gives

$$f(2a) + 2K = f(f(a)) = 2f(a) + K,$$

where the second equality follows from (2). Consequently,

$$f(2a) = 2f(a) - K \quad (3)$$

for all  $a \in \mathbb{Z}$ .

Substituting (2) and (3) into (1), we obtain

$$\begin{aligned} f(2a) + 2f(b) &= f(f(a + b)) \\ 2f(a) - K + 2f(b) &= 2f(a + b) + K \\ f(a) + f(b) &= f(a + b) + K. \end{aligned}$$

Thus, if we set  $g(n) = f(n) - K$  we see that  $g$  satisfies the Cauchy equation  $g(a + b) = g(a) + g(b)$ . The solution to the Cauchy equation over  $\mathbb{Z}$  is well-known; indeed, it may be proven by an easy induction that  $g(n) = Mn$  for each  $n \in \mathbb{Z}$ , where  $M = g(1)$  is a constant.

Therefore,  $f(n) = Mn + K$ , and we may proceed as in Solution 1.

**Comment 1.** Instead of deriving (3) by substituting  $b = 0$  into (1), we could instead have observed that the right hand side of (1) is symmetric in  $a$  and  $b$ , and thus

$$f(2a) + 2f(b) = f(2b) + 2f(a).$$

Thus,  $f(2a) - 2f(a) = f(2b) - 2f(b)$  for any  $a, b \in \mathbb{Z}$ , and in particular  $f(2a) - 2f(a)$  is constant. Setting  $a = 0$  shows that this constant is equal to  $-K$ , and so we obtain (3).

**Comment 2.** Some solutions initially prove that  $f(f(n))$  is linear (sometimes via proving that  $f(f(n)) - 3K$  satisfies the Cauchy equation). However, one can immediately prove that  $f$  is linear by substituting something of the form  $f(f(n)) = M'n + K'$  into (2).

**Problem 2.**

In triangle  $ABC$ , point  $A_1$  lies on side  $BC$  and point  $B_1$  lies on side  $AC$ . Let  $P$  and  $Q$  be points on segments  $AA_1$  and  $BB_1$ , respectively, such that  $PQ$  is parallel to  $AB$ . Let  $P_1$  be a point on line  $PB_1$ , such that  $B_1$  lies strictly between  $P$  and  $P_1$ , and  $\angle PP_1C = \angle BAC$ . Similarly, let  $Q_1$  be a point on line  $QA_1$ , such that  $A_1$  lies strictly between  $Q$  and  $Q_1$ , and  $\angle CQ_1Q = \angle CBA$ .

Prove that points  $P, Q, P_1$ , and  $Q_1$  are concyclic.

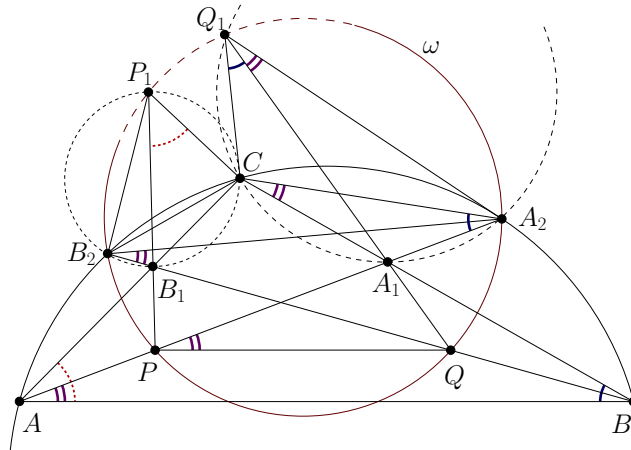
(Ukraine)

**Solution 1.** Throughout the solution we use oriented angles.

Let rays  $AA_1$  and  $BB_1$  intersect the circumcircle of  $\triangle ACB$  at  $A_2$  and  $B_2$ , respectively. By

$$\angle QPA_2 = \angle BAA_2 = \angle BB_2A_2 = \angle QB_2A_2,$$

points  $P, Q, A_2, B_2$  are concyclic; denote the circle passing through these points by  $\omega$ . We shall prove that  $P_1$  and  $Q_1$  also lie on  $\omega$ .



By

$$\angle CA_2A_1 = \angle CA_2A = \angle CBA = \angle CQ_1Q = \angle CQ_1A_1,$$

points  $C, Q_1, A_2, A_1$  are also concyclic. From that we get

$$\angle QQ_1A_2 = \angle A_1Q_1A_2 = \angle A_1CA_2 = \angle BCA_2 = \angle BAA_2 = \angle QPA_2,$$

so  $Q_1$  lies on  $\omega$ .

It follows similarly that  $P_1$  lies on  $\omega$ .

**Solution 2.** First consider the case when lines  $PP_1$  and  $QQ_1$  intersect each other at some point  $R$ .

Let line  $PQ$  meet the sides  $AC$  and  $BC$  at  $E$  and  $F$ , respectively. Then

$$\angle PP_1C = \angle BAC = \angle PEC,$$

so points  $C, E, P, P_1$  lie on a circle; denote that circle by  $\omega_P$ . It follows analogously that points  $C, F, Q, Q_1$  lie on another circle; denote it by  $\omega_Q$ .

Let  $AQ$  and  $BP$  intersect at  $T$ . Applying Pappus' theorem to the lines  $AA_1P$  and  $BB_1Q$  provides that points  $C = AB_1 \cap BA_1$ ,  $R = A_1Q \cap B_1P$  and  $T = AQ \cap BP$  are collinear.

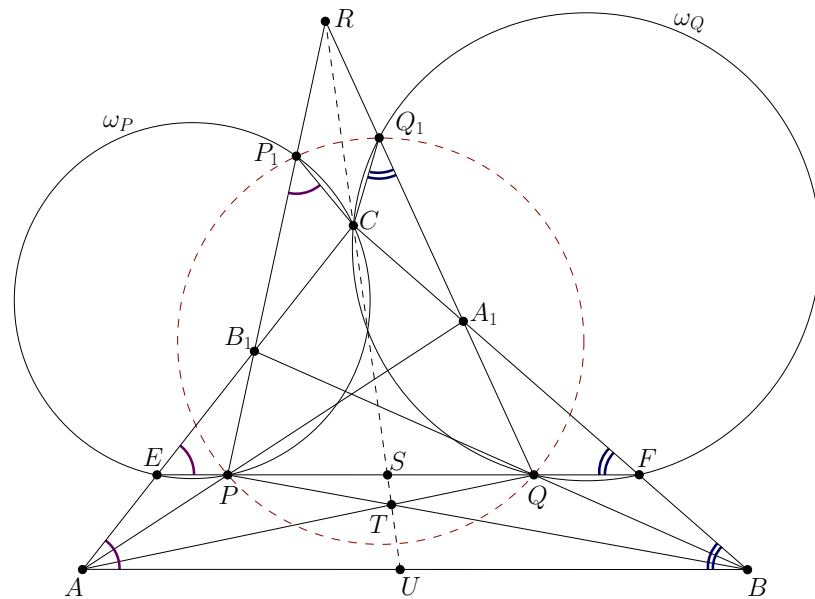
Let line  $RCT$  meet  $PQ$  and  $AB$  at  $S$  and  $U$ , respectively. From  $AB \parallel PQ$  we obtain

$$\frac{SP}{SQ} = \frac{UB}{UA} = \frac{SF}{SE},$$

so

$$SP \cdot SE = SQ \cdot SF.$$

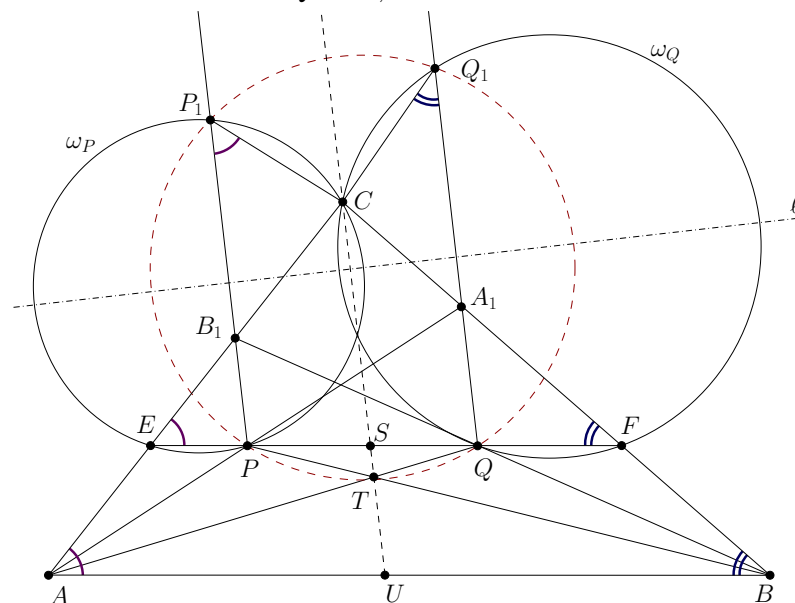




So, point  $S$  has equal powers with respect to  $\omega_P$  and  $\omega_Q$ , hence line  $RCS$  is their radical axis; then  $R$  also has equal powers to the circles, so  $RP \cdot RP_1 = RQ \cdot RQ_1$ , proving that points  $P, P_1, Q, Q_1$  are indeed concyclic.

Now consider the case when  $PP_1$  and  $QQ_1$  are parallel. Like in the previous case, let  $AQ$  and  $BP$  intersect at  $T$ . Applying Pappus' theorem again to the lines  $AA_1P$  and  $BB_1Q$ , in this limit case it shows that line  $CT$  is parallel to  $PP_1$  and  $QQ_1$ .

Let line  $CT$  meet  $PQ$  and  $AB$  at  $S$  and  $U$ , as before. The same calculation as in the previous case shows that  $SP \cdot SE = SQ \cdot SF$ , so  $S$  lies on the radical axis between  $\omega_P$  and  $\omega_Q$ .



Line  $CST$ , that is the radical axis between  $\omega_P$  and  $\omega_Q$ , is perpendicular to the line  $\ell$  of centres of  $\omega_P$  and  $\omega_Q$ . Hence, the chords  $PP_1$  and  $QQ_1$  are perpendicular to  $\ell$ . So the quadrilateral  $PP_1Q_1Q$  is an isosceles trapezium with symmetry axis  $\ell$ , and hence is cyclic.

**Comment.** There are several ways of solving the problem involving Pappus' theorem. For example, one may consider the points  $K = PB_1 \cap BC$  and  $L = QA_1 \cap AC$ . Applying Pappus' theorem to the lines  $AA_1P$  and  $QB_1B$  we get that  $K, L$ , and  $PQ \cap AB$  are collinear, i.e. that  $KL \parallel AB$ . Therefore, cyclicity of  $P, Q, P_1$ , and  $Q_1$  is equivalent to that of  $K, L, P_1$ , and  $Q_1$ . The latter is easy after noticing that  $C$  also lies on that circle. Indeed, e.g.  $\angle(LK, LC) = \angle(AB, AC) = \angle(P_1K, P_1C)$  shows that  $K$  lies on circle  $KLC$ .

This approach also has some possible degeneracy, as the points  $K$  and  $L$  may happen to be ideal.

**Problem 3.**

A social network has 2019 users, some pairs of whom are friends. Whenever user  $A$  is friends with user  $B$ , user  $B$  is also friends with user  $A$ . Events of the following kind may happen repeatedly, one at a time:

Three users  $A$ ,  $B$ , and  $C$  such that  $A$  is friends with both  $B$  and  $C$ , but  $B$  and  $C$  are not friends, change their friendship statuses such that  $B$  and  $C$  are now friends, but  $A$  is no longer friends with  $B$ , and no longer friends with  $C$ . All other friendship statuses are unchanged.

Initially, 1010 users have 1009 friends each, and 1009 users have 1010 friends each. Prove that there exists a sequence of such events after which each user is friends with at most one other user.

(Croatia)

**Common remarks.** The problem has an obvious rephrasing in terms of graph theory. One is given a graph  $G$  with 2019 vertices, 1010 of which have degree 1009 and 1009 of which have degree 1010. One is allowed to perform operations on  $G$  of the following kind:

Suppose that vertex  $A$  is adjacent to two distinct vertices  $B$  and  $C$  which are not adjacent to each other. Then one may remove the edges  $AB$  and  $AC$  from  $G$  and add the edge  $BC$  into  $G$ .

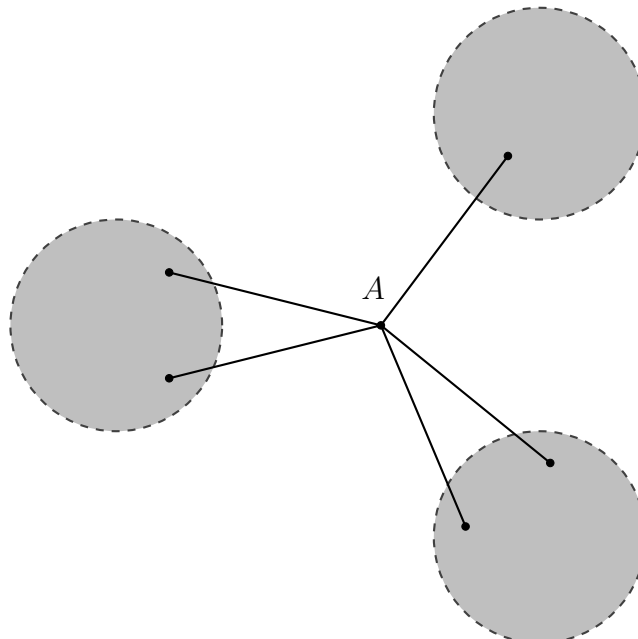
Call such an operation a *refriending*. One wants to prove that, via a sequence of such refriendings, one can reach a graph which is a disjoint union of single edges and vertices.

All of the solutions presented below will use this reformulation.

**Solution 1.** Note that the given graph is connected, since the total degree of any two vertices is at least 2018 and hence they are either adjacent or have at least one neighbour in common. Hence the given graph satisfies the following condition:

Every connected component of  $G$  with at least three vertices is not complete and has a vertex of odd degree. (1)

We will show that if a graph  $G$  satisfies condition (1) and has a vertex of degree at least 2, then there is a refriending on  $G$  that preserves condition (1). Since refriendings decrease the total number of edges of  $G$ , by using a sequence of such refriendings, we must reach a graph  $G$  with maximal degree at most 1, so we are done.



Pick a vertex  $A$  of degree at least 2 in a connected component  $G'$  of  $G$ . Since no component of  $G$  with at least three vertices is complete we may assume that not all of the neighbours of  $A$  are adjacent to one another. (For example, pick a maximal complete subgraph  $K$  of  $G'$ . Some vertex  $A$  of  $K$  has a neighbour outside  $K$ , and this neighbour is not adjacent to every vertex of  $K$  by maximality.) Removing  $A$  from  $G$  splits  $G'$  into smaller connected components  $G_1, \dots, G_k$  (possibly with  $k = 1$ ), to each of which  $A$  is connected by at least one edge. We divide into several cases.

*Case 1:  $k \geq 2$  and  $A$  is connected to some  $G_i$  by at least two edges.*

Choose a vertex  $B$  of  $G_i$  adjacent to  $A$ , and a vertex  $C$  in another component  $G_j$  adjacent to  $A$ . The vertices  $B$  and  $C$  are not adjacent, and hence removing edges  $AB$  and  $AC$  and adding in edge  $BC$  does not disconnect  $G'$ . It is easy to see that this preserves the condition, since the refriending does not change the parity of the degrees of vertices.

*Case 2:  $k \geq 2$  and  $A$  is connected to each  $G_i$  by exactly one edge.*

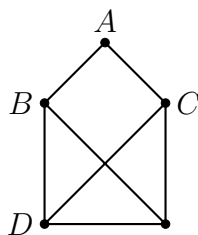
Consider the induced subgraph on any  $G_i$  and the vertex  $A$ . The vertex  $A$  has degree 1 in this subgraph; since the number of odd-degree vertices of a graph is always even, we see that  $G_i$  has a vertex of odd degree (in  $G$ ). Thus if we let  $B$  and  $C$  be any distinct neighbours of  $A$ , then removing edges  $AB$  and  $AC$  and adding in edge  $BC$  preserves the above condition: the refriending creates two new components, and if either of these components has at least three vertices, then it cannot be complete and must contain a vertex of odd degree (since each  $G_i$  does).

*Case 3:  $k = 1$  and  $A$  is connected to  $G_1$  by at least three edges.*

By assumption,  $A$  has two neighbours  $B$  and  $C$  which are not adjacent to one another. Removing edges  $AB$  and  $AC$  and adding in edge  $BC$  does not disconnect  $G'$ . We are then done as in Case 1.

*Case 4:  $k = 1$  and  $A$  is connected to  $G_1$  by exactly two edges.*

Let  $B$  and  $C$  be the two neighbours of  $A$ , which are not adjacent. Removing edges  $AB$  and  $AC$  and adding in edge  $BC$  results in two new components: one consisting of a single vertex; and the other containing a vertex of odd degree. We are done unless this second component would be a complete graph on at least 3 vertices. But in this case,  $G_1$  would be a complete graph minus the single edge  $BC$ , and hence has at least 4 vertices since  $G'$  is not a 4-cycle. If we let  $D$  be a third vertex of  $G_1$ , then removing edges  $BA$  and  $BD$  and adding in edge  $AD$  does not disconnect  $G'$ . We are then done as in Case 1.

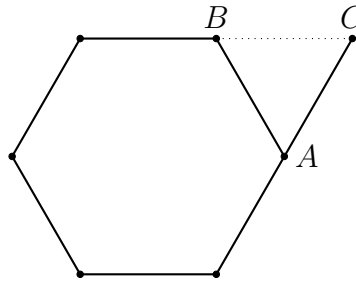


**Comment.** In fact, condition 1 above *precisely* characterises those graphs which can be reduced to a graph of maximal degree  $\leq 1$  by a sequence of refriendings.

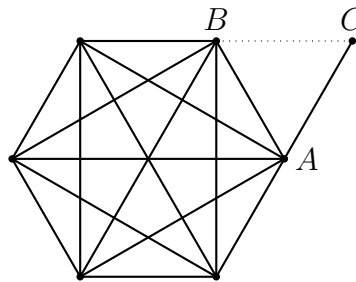
**Solution 2.** As in the previous solution, note that a refriending preserves the property that a graph has a vertex of odd degree and (trivially) the property that it is not complete; note also that our initial graph is connected. We describe an algorithm to reduce our initial graph to a graph of maximal degree at most 1, proceeding in two steps.

*Step 1: There exists a sequence of refriendings reducing the graph to a tree.*

*Proof.* Since the number of edges decreases with each refriending, it suffices to prove the following: as long as the graph contains a cycle, there exists a refriending such that the resulting graph is still connected. We will show that the graph in fact contains a cycle  $Z$  and vertices  $A, B, C$  such that  $A$  and  $B$  are adjacent in the cycle  $Z$ ,  $C$  is not in  $Z$ , and is adjacent to  $A$  but not  $B$ . Removing edges  $AB$  and  $AC$  and adding in edge  $BC$  keeps the graph connected, so we are done.



To find this cycle  $Z$  and vertices  $A, B, C$ , we pursue one of two strategies. If the graph contains a triangle, we consider a largest complete subgraph  $K$ , which thus contains at least three vertices. Since the graph itself is not complete, there is a vertex  $C$  not in  $K$  connected to a vertex  $A$  of  $K$ . By maximality of  $K$ , there is a vertex  $B$  of  $K$  not connected to  $C$ , and hence we are done by choosing a cycle  $Z$  in  $K$  through the edge  $AB$ .



If the graph is triangle-free, we consider instead a smallest cycle  $Z$ . This cycle cannot be Hamiltonian (i.e. it cannot pass through every vertex of the graph), since otherwise by minimality the graph would then have no other edges, and hence would have even degree at every vertex. We may thus choose a vertex  $C$  not in  $Z$  adjacent to a vertex  $A$  of  $Z$ . Since the graph is triangle-free, it is not adjacent to any neighbour  $B$  of  $A$  in  $Z$ , and we are done.  $\square$

*Step 2: Any tree may be reduced to a disjoint union of single edges and vertices by a sequence of refriendings.*

*Proof.* The refriending preserves the property of being acyclic. Hence, after applying a sequence of refriendings, we arrive at an acyclic graph in which it is impossible to perform any further refriendings. The maximal degree of any such graph is 1: if it had a vertex  $A$  with two neighbours  $B, C$ , then  $B$  and  $C$  would necessarily be nonadjacent since the graph is cycle-free, and so a refriending would be possible. Thus we reach a graph with maximal degree at most 1 as desired.  $\square$

## Day 2

### Problem 4.

Find all pairs  $(k, n)$  of positive integers such that

$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}). \quad (1)$$

(El Salvador)

**Answer:** The only such pairs are  $(1, 1)$  and  $(3, 2)$ .

**Common remarks.** In all solutions, for any prime  $p$  and positive integer  $N$ , we will denote by  $v_p(N)$  the exponent of the largest power of  $p$  that divides  $N$ . The right-hand side of (1) will be denoted by  $L_n$ ; that is,  $L_n = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1})$ .

**Solution 1.** We will get an upper bound on  $n$  from the speed at which  $v_2(L_n)$  grows.

From

$$L_n = (2^n - 1)(2^n - 2) \cdots (2^n - 2^{n-1}) = 2^{1+2+\cdots+(n-1)}(2^n - 1)(2^{n-1} - 1) \cdots (2^1 - 1)$$

we read

$$v_2(L_n) = 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}.$$

On the other hand,  $v_2(k!)$  is expressed by the Legendre formula as

$$v_2(k!) = \sum_{i=1}^{\infty} \left\lfloor \frac{k}{2^i} \right\rfloor.$$

As usual, by omitting the floor functions,

$$v_2(k!) < \sum_{i=1}^{\infty} \frac{k}{2^i} = k.$$

Thus,  $k! = L_n$  implies the inequality

$$\frac{n(n-1)}{2} < k. \quad (2)$$

In order to obtain an opposite estimate, observe that

$$L_n = (2^n - 1)(2^n - 2) \cdots (2^n - 2^{n-1}) < (2^n)^n = 2^{n^2}.$$

We claim that

$$2^{n^2} < \left( \frac{n(n-1)}{2} \right)! \quad \text{for } n \geq 6. \quad (3)$$

For  $n = 6$  the estimate (3) is true because  $2^{6^2} < 6.9 \cdot 10^{10}$  and  $\left( \frac{6(6-1)}{2} \right)! = 15! > 1.3 \cdot 10^{12}$ .

For  $n \geq 7$  we prove (3) by the following inequalities:

$$\begin{aligned} \left( \frac{n(n-1)}{2} \right)! &= 15! \cdot 16 \cdot 17 \cdots \frac{n(n-1)}{2} > 2^{36} \cdot 16^{\frac{n(n-1)}{2}-15} \\ &= 2^{2n(n-1)-24} = 2^{n^2} \cdot 2^{n(n-2)-24} > 2^{n^2}. \end{aligned}$$

Putting together (2) and (3), for  $n \geq 6$  we get a contradiction, since

$$L_n < 2^{n^2} < \left(\frac{n(n-1)}{2}\right)! < k! = L_n.$$

Hence  $n \geq 6$  is not possible.

Checking manually the cases  $n \leq 5$  we find

$$\begin{aligned} L_1 = 1 = 1!, \quad L_2 = 6 = 3!, \quad 5! < L_3 = 168 < 6!, \\ 7! < L_4 = 20\,160 < 8! \quad \text{and} \quad 10! < L_5 = 9\,999\,360 < 11!. \end{aligned}$$

So, there are two solutions:

$$(k, n) \in \{(1, 1), (3, 2)\}.$$

**Solution 2.** Like in the previous solution, the cases  $n = 1, 2, 3, 4$  are checked manually. We will exclude  $n \geq 5$  by considering the exponents of 3 and 31 in (1).

For odd primes  $p$  and distinct integers  $a, b$ , coprime to  $p$ , with  $p \mid a - b$ , the Lifting The Exponent lemma asserts that

$$v_p(a^j - b^j) = v_p(a - b) + v_p(j).$$

Notice that 3 divides  $2^j - 1$  if only if  $j$  is even; moreover, by the Lifting The Exponent lemma we have

$$v_3(2^{2j} - 1) = v_3(4^j - 1) = 1 + v_3(j) = v_3(3j).$$

Hence,

$$v_3(L_n) = \sum_{2^j \leq n} v_3(4^j - 1) = \sum_{j \leq \lfloor \frac{n}{2} \rfloor} v_3(3j).$$

Notice that the last expression is precisely the exponent of 3 in the prime factorisation of  $(3\lfloor \frac{n}{2} \rfloor)!$ . Therefore

$$\begin{aligned} v_3(k!) = v_3(L_n) &= v_3\left(\left(3\left\lfloor \frac{n}{2} \right\rfloor\right)!\right) \\ 3\left\lfloor \frac{n}{2} \right\rfloor &\leq k \leq 3\left\lfloor \frac{n}{2} \right\rfloor + 2. \end{aligned} \tag{4}$$

Suppose that  $n \geq 5$ . Note that every fifth factor in  $L_n$  is divisible by  $31 = 2^5 - 1$ , and hence we have  $v_{31}(L_n) \geq \lfloor \frac{n}{5} \rfloor$ . Then

$$\frac{n}{10} \leq \left\lfloor \frac{n}{5} \right\rfloor \leq v_{31}(L_n) = v_{31}(k!) = \sum_{j=1}^{\infty} \left\lfloor \frac{k}{31^j} \right\rfloor < \sum_{j=1}^{\infty} \frac{k}{31^j} = \frac{k}{30}. \tag{5}$$

By combining (4) and (5),

$$3n < k \leq \frac{3n}{2} + 2$$

so  $n < \frac{4}{3}$  which is inconsistent with the inequality  $n \geq 5$ .

**Comment 1.** There are many combinations of the ideas above; for example combining (2) and (4) also provides  $n < 5$ . Obviously, considering the exponents of any two primes in (1), or considering one prime and the magnitude orders lead to an upper bound on  $n$  and  $k$ .

**Comment 2.** This problem has a connection to group theory. Indeed, the right-hand side is the order of the group  $GL_n(\mathbb{F}_2)$  of invertible  $n$ -by- $n$  matrices with entries modulo 2, while the left-hand side is the order of the symmetric group  $S_k$  on  $k$  elements. The result thus shows that the only possible isomorphisms between these groups are  $GL_1(\mathbb{F}_2) \cong S_1$  and  $GL_2(\mathbb{F}_2) \cong S_3$ , and there are in fact isomorphisms in both cases. In general,  $GL_n(\mathbb{F}_2)$  is a simple group for  $n \geq 3$ , as it is isomorphic to  $PSL_n(\mathbb{F}_2)$ .

There is also a near-solution of interest: the right-hand side for  $n = 4$  is *half* of the left-hand side when  $k = 8$ ; this turns out to correspond to an isomorphism  $GL_4(\mathbb{F}_2) \cong A_8$  with the alternating group on eight elements.

However, while this indicates that the problem is a useful one, knowing group theory is of no use in solving it!

**Problem 5.**

The Bank of Bath issues coins with an  $H$  on one side and a  $T$  on the other. Harry has  $n$  of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly  $k > 0$  coins showing  $H$ , then he turns over the  $k^{\text{th}}$  coin from the left; otherwise, all coins show  $T$  and he stops. For example, if  $n = 3$  the process starting with the configuration  $THT$  would be  $THT \rightarrow HHT \rightarrow HTT \rightarrow TTT$ , which stops after three operations.

- (a) Show that, for each initial configuration, Harry stops after a finite number of operations.
- (b) For each initial configuration  $C$ , let  $L(C)$  be the number of operations before Harry stops. For example,  $L(THT) = 3$  and  $L(TTT) = 0$ . Determine the average value of  $L(C)$  over all  $2^n$  possible initial configurations  $C$ .

(USA)

**Answer:** The average is  $\frac{1}{4}n(n+1)$ .

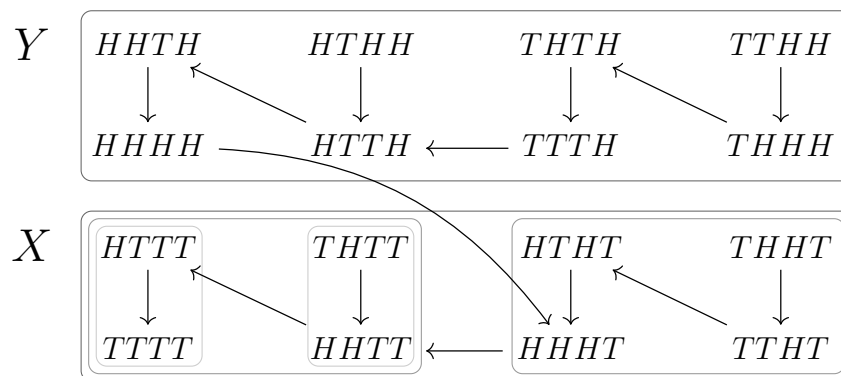
**Common remarks.** Throughout all these solutions, we let  $E(n)$  denote the desired average value.

**Solution 1.** We represent the problem using a directed graph  $G_n$  whose vertices are the length- $n$  strings of  $H$ 's and  $T$ 's. The graph features an edge from each string to its successor (except for  $TT \cdots TT$ , which has no successor). We will also write  $\bar{H} = T$  and  $\bar{T} = H$ .

The graph  $G_0$  consists of a single vertex: the empty string. The main claim is that  $G_n$  can be described explicitly in terms of  $G_{n-1}$ :

- We take two copies,  $X$  and  $Y$ , of  $G_{n-1}$ .
- In  $X$ , we take each string of  $n-1$  coins and just append a  $T$  to it. In symbols, we replace  $s_1 \cdots s_{n-1}$  with  $s_1 \cdots s_{n-1}T$ .
- In  $Y$ , we take each string of  $n-1$  coins, flip every coin, reverse the order, and append an  $H$  to it. In symbols, we replace  $s_1 \cdots s_{n-1}$  with  $\bar{s}_{n-1}\bar{s}_{n-2} \cdots \bar{s}_1H$ .
- Finally, we add one new edge from  $Y$  to  $X$ , namely  $HH \cdots HHH \rightarrow HH \cdots HHT$ .

We depict  $G_4$  below, in a way which indicates this recursive construction:



We prove the claim inductively. Firstly,  $X$  is correct as a subgraph of  $G_n$ , as the operation on coins is unchanged by an extra  $T$  at the end: if  $s_1 \cdots s_{n-1}$  is sent to  $t_1 \cdots t_{n-1}$ , then  $s_1 \cdots s_{n-1}T$  is sent to  $t_1 \cdots t_{n-1}T$ .

Next,  $Y$  is also correct as a subgraph of  $G_n$ , as if  $s_1 \cdots s_{n-1}$  has  $k$  occurrences of  $H$ , then  $\bar{s}_{n-1} \cdots \bar{s}_1H$  has  $(n-1-k) + 1 = n-k$  occurrences of  $H$ , and thus (provided that  $k > 0$ ), if  $s_1 \cdots s_{n-1}$  is sent to  $t_1 \cdots t_{n-1}$ , then  $\bar{s}_{n-1} \cdots \bar{s}_1H$  is sent to  $\bar{t}_{n-1} \cdots \bar{t}_1H$ .



Finally, the one edge from  $Y$  to  $X$  is correct, as the operation does send  $HH \cdots HHH$  to  $HH \cdots HHT$ .

To finish, note that the sequences in  $X$  take an average of  $E(n-1)$  steps to terminate, whereas the sequences in  $Y$  take an average of  $E(n-1)$  steps to reach  $HH \cdots H$  and then an additional  $n$  steps to terminate. Therefore, we have

$$E(n) = \frac{1}{2} (E(n-1) + (E(n-1) + n)) = E(n-1) + \frac{n}{2}.$$

We have  $E(0) = 0$  from our description of  $G_0$ . Thus, by induction, we have  $E(n) = \frac{1}{2}(1 + \cdots + n) = \frac{1}{4}n(n+1)$ , which in particular is finite.

**Solution 2.** We consider what happens with configurations depending on the coins they start and end with.

- If a configuration starts with  $H$ , the last  $n-1$  coins follow the given rules, as if they were all the coins, until they are all  $T$ , then the first coin is turned over.
- If a configuration ends with  $T$ , the last coin will never be turned over, and the first  $n-1$  coins follow the given rules, as if they were all the coins.
- If a configuration starts with  $T$  and ends with  $H$ , the middle  $n-2$  coins follow the given rules, as if they were all the coins, until they are all  $T$ . After that, there are  $2n-1$  more steps: first coins  $1, 2, \dots, n-1$  are turned over in that order, then coins  $n, n-1, \dots, 1$  are turned over in that order.

As this covers all configurations, and the number of steps is clearly finite for 0 or 1 coins, it follows by induction on  $n$  that the number of steps is always finite.

We define  $E_{AB}(n)$ , where  $A$  and  $B$  are each one of  $H, T$  or  $*$ , to be the average number of steps over configurations of length  $n$  restricted to those that start with  $A$ , if  $A$  is not  $*$ , and that end with  $B$ , if  $B$  is not  $*$  (so  $*$  represents “either  $H$  or  $T$ ”). The above observations tell us that, for  $n \geq 2$ :

- $E_{H*}(n) = E(n-1) + 1$ .
- $E_{*T}(n) = E(n-1)$ .
- $E_{HT}(n) = E(n-2) + 1$  (by using both the observations for  $H*$  and for  $*T$ ).
- $E_{TH}(n) = E(n-2) + 2n - 1$ .

Now  $E_{H*}(n) = \frac{1}{2}(E_{HH}(n) + E_{HT}(n))$ , so  $E_{HH}(n) = 2E(n-1) - E(n-2) + 1$ . Similarly,  $E_{TT}(n) = 2E(n-1) - E(n-2) - 1$ . So

$$E(n) = \frac{1}{4}(E_{HT}(n) + E_{HH}(n) + E_{TT}(n) + E_{TH}(n)) = E(n-1) + \frac{n}{2}.$$

We have  $E(0) = 0$  and  $E(1) = \frac{1}{2}$ , so by induction on  $n$  we have  $E(n) = \frac{1}{4}n(n+1)$ .

**Solution 3.** Let  $H_i$  be the number of  $H$ 's in positions 1 to  $i$  inclusive (so  $H_n$  is the total number of  $H$ 's), and let  $I_i$  be 1 if the  $i^{\text{th}}$  coin is an  $H$ , 0 otherwise. Consider the function

$$t(i) = I_i + 2(\min\{i, H_n\} - H_i).$$

We claim that  $t(i)$  is the total number of times coin  $i$  is turned over (which implies that the process terminates). Certainly  $t(i) = 0$  when all coins are  $T$ 's, and  $t(i)$  is always a nonnegative integer, so it suffices to show that when the  $k^{\text{th}}$  coin is turned over (where  $k = H_n$ ),  $t(k)$  goes down by 1 and all the other  $t(i)$  are unchanged. We show this by splitting into cases:

- If  $i < k$ ,  $I_i$  and  $H_i$  are unchanged, and  $\min\{i, H_n\} = i$  both before and after the coin flip, so  $t(i)$  is unchanged.
- If  $i > k$ ,  $\min\{i, H_n\} = H_n$  both before and after the coin flip, and both  $H_n$  and  $H_i$  change by the same amount, so  $t(i)$  is unchanged.
- If  $i = k$  and the coin is  $H$ ,  $I_i$  goes down by 1, as do both  $\min\{i, H_n\} = H_n$  and  $H_i$ ; so  $t(i)$  goes down by 1.
- If  $i = k$  and the coin is  $T$ ,  $I_i$  goes up by 1,  $\min\{i, H_n\} = i$  is unchanged and  $H_i$  goes up by 1; so  $t(i)$  goes down by 1.

We now need to compute the average value of

$$\sum_{i=1}^n t(i) = \sum_{i=1}^n I_i + 2 \sum_{i=1}^n \min\{i, H_n\} - 2 \sum_{i=1}^n H_i.$$

The average value of the first term is  $\frac{1}{2}n$ , and that of the third term is  $-\frac{1}{2}n(n+1)$ . To compute the second term, we sum over choices for the total number of  $H$ 's, and then over the possible values of  $i$ , getting

$$2^{1-n} \sum_{j=0}^n \binom{n}{j} \sum_{i=1}^n \min\{i, j\} = 2^{1-n} \sum_{j=0}^n \binom{n}{j} \left( nj - \binom{j}{2} \right).$$

Now, in terms of trinomial coefficients,

$$\sum_{j=0}^n j \binom{n}{j} = \sum_{j=1}^n \binom{n}{n-j, j-1, 1} = n \sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1}n$$

and

$$\sum_{j=0}^n \binom{j}{2} \binom{n}{j} = \sum_{j=2}^n \binom{n}{n-j, j-2, 2} = \binom{n}{2} \sum_{j=0}^{n-2} \binom{n-2}{j} = 2^{n-2} \binom{n}{2}.$$

So the second term above is

$$2^{1-n} \left( 2^{n-1}n^2 - 2^{n-2} \binom{n}{2} \right) = n^2 - \frac{n(n-1)}{4},$$

and the required average is

$$E(n) = \frac{1}{2}n + n^2 - \frac{n(n-1)}{4} - \frac{1}{2}n(n+1) = \frac{n(n+1)}{4}.$$

**Solution 4.** Harry has built a Turing machine to flip the coins for him. The machine is initially positioned at the  $k^{\text{th}}$  coin, where there are  $k$  coins showing  $H$  (and the position before the first coin is considered to be the  $0^{\text{th}}$  coin). The machine then moves according to the following rules, stopping when it reaches the position before the first coin: if the coin at its current position is  $H$ , it flips the coin and moves to the previous coin, while if the coin at its current position is  $T$ , it flips the coin and moves to the next position.

Consider the maximal sequences of consecutive moves in the same direction. Suppose the machine has  $a$  consecutive moves to the next coin, before a move to the previous coin. After those  $a$  moves, the  $a$  coins flipped in those moves are all  $H$ 's, as is the coin the machine is now at, so at least the next  $a+1$  moves will all be moves to the previous coin. Similarly,  $a$  consecutive moves to the previous coin are followed by at least  $a+1$  consecutive moves to

the next coin. There cannot be more than  $n$  consecutive moves in the same direction, so this proves that the process terminates (with a move from the first coin to the position before the first coin).

Thus we have a (possibly empty) sequence  $a_1 < \dots < a_t \leq n$  giving the lengths of maximal sequences of consecutive moves in the same direction, where the final  $a_t$  moves must be moves to the previous coin, ending before the first coin. We claim there is a bijection between initial configurations of the coins and such sequences. This gives

$$E(n) = \frac{1}{2}(1 + 2 + \dots + n) = \frac{n(n+1)}{4}$$

as required, since each  $i$  with  $1 \leq i \leq n$  will appear in half of the sequences, and will contribute  $i$  to the number of moves when it does.

To see the bijection, consider following the sequence of moves backwards, starting with the machine just before the first coin and all coins showing  $T$ . This certainly determines a unique configuration of coins that could possibly correspond to the given sequence. Furthermore, every coin flipped as part of the  $a_j$  consecutive moves is also flipped as part of all subsequent sequences of  $a_k$  consecutive moves, for all  $k > j$ , meaning that, as we follow the moves backwards, each coin is always in the correct state when flipped to result in a move in the required direction. (Alternatively, since there are  $2^n$  possible configurations of coins and  $2^n$  possible such ascending sequences, the fact that the sequence of moves determines at most one configuration of coins, and thus that there is an injection from configurations of coins to such ascending sequences, is sufficient for it to be a bijection, without needing to show that coins are in the right state as we move backwards.)

**Solution 5.** We explicitly describe what happens with an arbitrary sequence  $C$  of  $n$  coins. Suppose that  $C$  contain  $k$  coins showing  $H$  at positions  $1 \leq c_1 < c_2 < \dots < c_k \leq n$ .

Let  $i$  be the minimal index such that  $c_i \geq k$ . Then the first few steps will consist of turning over the  $k^{\text{th}}$ ,  $(k+1)^{\text{th}}$ ,  $\dots$ ,  $c_i^{\text{th}}$ ,  $(c_i-1)^{\text{th}}$ ,  $(c_i-2)^{\text{th}}$ ,  $\dots$ ,  $k^{\text{th}}$  coins in this order. After that we get a configuration with  $k-1$  coins showing  $H$  at the same positions as in the initial one, except for  $c_i$ . This part of the process takes  $2(c_i - k) + 1$  steps.

After that, the process acts similarly; by induction on the number of  $H$ 's we deduce that the process ends. Moreover, if the  $c_i$  disappear in order  $c_{i_1}, \dots, c_{i_k}$ , the whole process takes

$$L(C) = \sum_{j=1}^k (2(c_{i_j} - (k+1-j)) + 1) = 2 \sum_{j=1}^k c_j - 2 \sum_{j=1}^k (k+1-j) + k = 2 \sum_{j=1}^k c_j - k^2$$

steps.

Now let us find the total value  $S_k$  of  $L(C)$  over all  $\binom{n}{k}$  configurations with exactly  $k$  coins showing  $H$ . To sum up the above expression over those, notice that each number  $1 \leq i \leq n$  appears as  $c_j$  exactly  $\binom{n-1}{k-1}$  times. Thus

$$\begin{aligned} S_k &= 2 \binom{n-1}{k-1} \sum_{i=1}^n i - \binom{n}{k} k^2 = 2 \frac{(n-1) \cdots (n-k+1)}{(k-1)!} \cdot \frac{n(n+1)}{2} - \frac{n \cdots (n-k+1)}{k!} k^2 \\ &= \frac{n(n-1) \cdots (n-k+1)}{(k-1)!} ((n+1) - k) = n(n-1) \binom{n-2}{k-1} + n \binom{n-1}{k-1}. \end{aligned}$$

Therefore, the total value of  $L(C)$  over all configurations is

$$\sum_{k=1}^n S_k = n(n-1) \sum_{k=1}^n \binom{n-2}{k-1} + n \sum_{k=1}^n \binom{n-1}{k-1} = n(n-1)2^{n-2} + n2^{n-1} = 2^n \frac{n(n+1)}{4}.$$

Hence the required average is  $E(n) = \frac{n(n+1)}{4}$ .

**Problem 6.**

Let  $I$  be the incentre of acute triangle  $ABC$  with  $AB \neq AC$ . The incircle  $\omega$  of  $ABC$  is tangent to sides  $BC$ ,  $CA$ , and  $AB$  at  $D$ ,  $E$ , and  $F$ , respectively. The line through  $D$  perpendicular to  $EF$  meets  $\omega$  again at  $R$ . Line  $AR$  meets  $\omega$  again at  $P$ . The circumcircles of triangles  $PCE$  and  $PBF$  meet again at  $Q$ .

Prove that lines  $DI$  and  $PQ$  meet on the line through  $A$  perpendicular to  $AI$ .

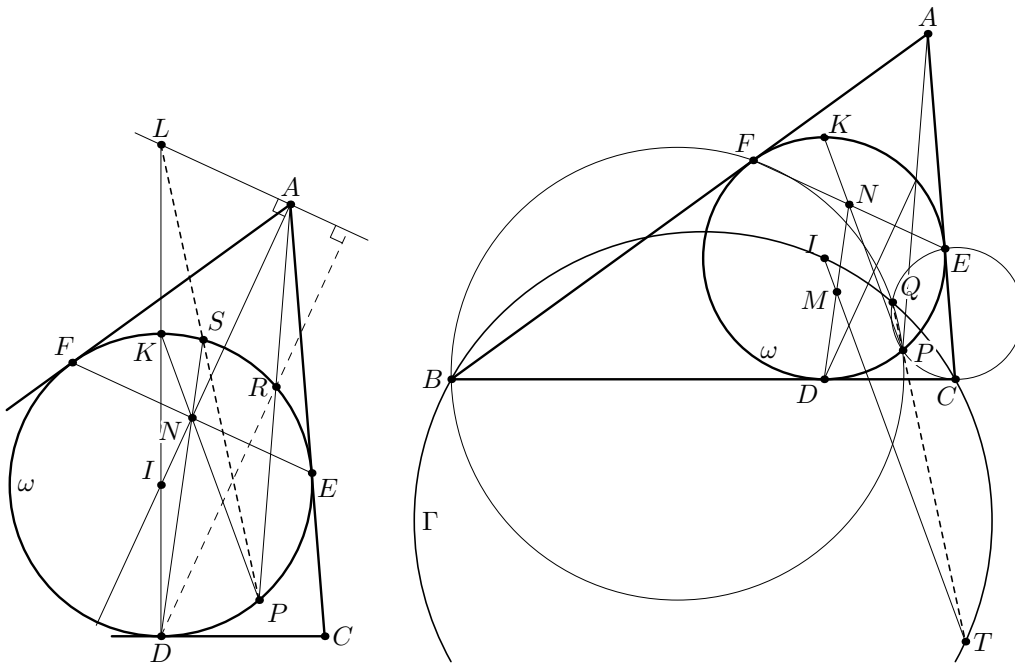
(India)

**Common remarks.** Throughout the solution,  $\angle(a, b)$  denotes the directed angle between lines  $a$  and  $b$ , measured modulo  $\pi$ .

**Solution 1.**

*Step 1.* The external bisector of  $\angle BAC$  is the line through  $A$  perpendicular to  $IA$ . Let  $DI$  meet this line at  $L$  and let  $DI$  meet  $\omega$  at  $K$ . Let  $N$  be the midpoint of  $EF$ , which lies on  $IA$  and is the pole of line  $AL$  with respect to  $\omega$ . Since  $AN \cdot AI = AE^2 = AR \cdot AP$ , the points  $R$ ,  $N$ ,  $I$ , and  $P$  are concyclic. As  $IR = IP$ , the line  $NI$  is the external bisector of  $\angle PNR$ , so  $PN$  meets  $\omega$  again at the point symmetric to  $R$  with respect to  $AN$  – i.e. at  $K$ .

Let  $DN$  cross  $\omega$  again at  $S$ . Opposite sides of any quadrilateral inscribed in the circle  $\omega$  meet on the polar line of the intersection of the diagonals with respect to  $\omega$ . Since  $L$  lies on the polar line  $AL$  of  $N$  with respect to  $\omega$ , the line  $PS$  must pass through  $L$ . Thus it suffices to prove that the points  $S$ ,  $Q$ , and  $P$  are collinear.



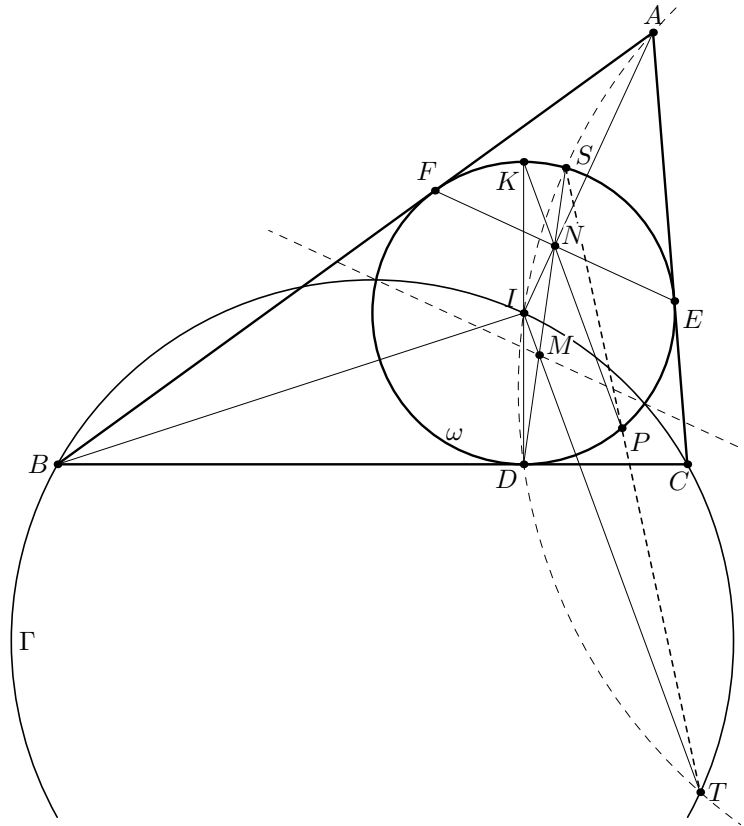
*Step 2.* Let  $\Gamma$  be the circumcircle of  $\triangle BIC$ . Notice that

$$\begin{aligned} \angle(BQ, QC) &= \angle(BQ, QP) + \angle(PQ, QC) = \angle(BF, FP) + \angle(PE, EC) \\ &= \angle(EF, EP) + \angle(FP, FE) = \angle(FP, EP) = \angle(DF, DE) = \angle(BI, IC), \end{aligned}$$

so  $Q$  lies on  $\Gamma$ . Let  $QP$  meet  $\Gamma$  again at  $T$ . It will now suffice to prove that  $S$ ,  $P$ , and  $T$  are collinear. Notice that  $\angle(BI, IT) = \angle(BQ, QT) = \angle(BF, FP) = \angle(FK, KP)$ . Note  $FD \perp FK$  and  $FD \perp BI$  so  $FK \parallel BI$  and hence  $IT$  is parallel to the line  $KNP$ . Since  $DI = IK$ , the line  $IT$  crosses  $DN$  at its midpoint  $M$ .

*Step 3.* Let  $F'$  and  $E'$  be the midpoints of  $DE$  and  $DF$ , respectively. Since  $DE' \cdot E'F = DE'^2 = BE' \cdot E'I$ , the point  $E'$  lies on the radical axis of  $\omega$  and  $\Gamma$ ; the same holds for  $F'$ . Therefore, this

radical axis is  $E'F'$ , and it passes through  $M$ . Thus  $IM \cdot MT = DM \cdot MS$ , so  $S, I, D$ , and  $T$  are concyclic. This shows  $\angle(DS, ST) = \angle(DI, IT) = \angle(DK, KP) = \angle(DS, SP)$ , whence the points  $S, P$ , and  $T$  are collinear, as desired.



**Comment.** Here is a longer alternative proof in step 1 that  $P, S$ , and  $L$  are collinear, using a circular inversion instead of the fact that opposite sides of a quadrilateral inscribed in a circle  $\omega$  meet on the polar line with respect to  $\omega$  of the intersection of the diagonals. Let  $G$  be the foot of the altitude from  $N$  to the line  $DIKL$ . Observe that  $N, G, K, S$  are concyclic (opposite right angles) so

$$\angle DIP = 2\angle DKP = \angle GKN + \angle DSP = \angle GSN + \angle NSP = \angle GSP,$$

hence  $I, G, S, P$  are concyclic. We have  $IG \cdot IL = IN \cdot IA = r^2$  since  $\triangle IGN \sim \triangle IAL$ . Inverting the circle  $IGSP$  in circle  $\omega$ , points  $P$  and  $S$  are fixed and  $G$  is taken to  $L$  so we find that  $P, S$ , and  $L$  are collinear.



Similarly, if  $V$  and  $Z$  are the second meeting points of  $\Omega_C$  with  $AR$  and  $EF$ , we get that the 4-tuples  $(C, V, I, X)$  and  $(B, I, Z, W)$  are both concyclic.

*Step 3.* Let  $Q' = CY \cap BZ$ . We will show that  $Q' = Q$ .

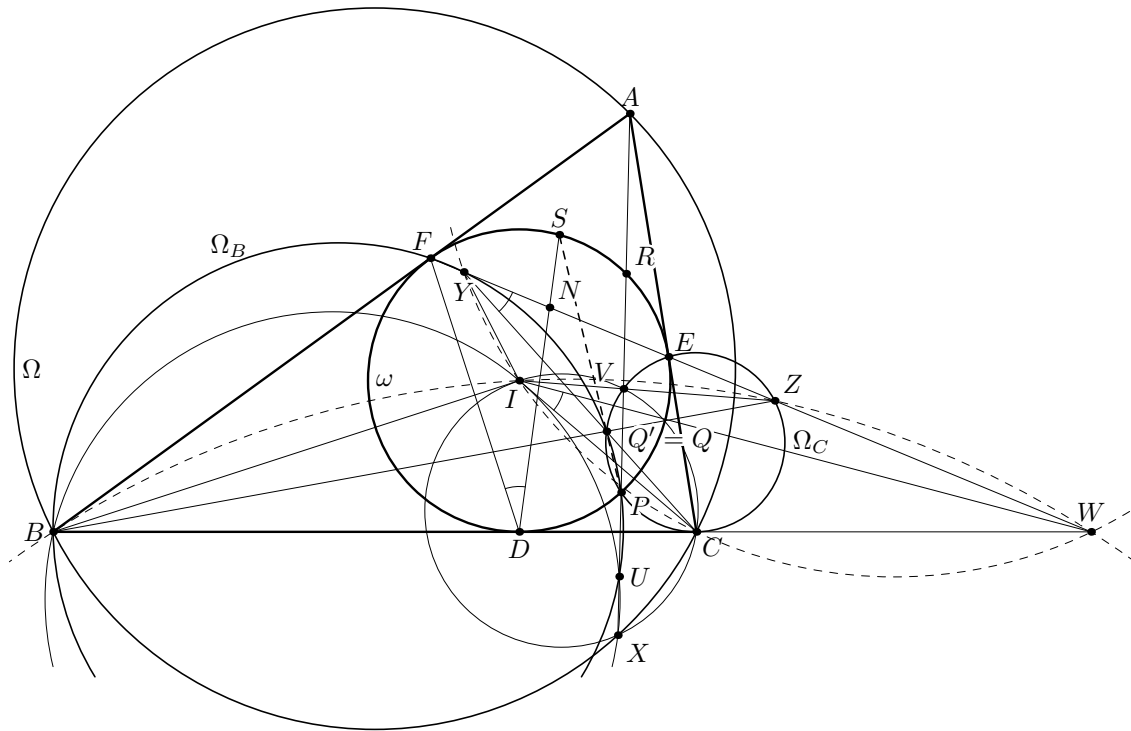
First of all, we have

$$\begin{aligned} \angle(Q'Y, Q'B) &= \angle(CY, ZB) = \angle(CY, ZY) + \angle(ZY, BZ) \\ &= \angle(CI, IW) + \angle(IW, IB) = \angle(CI, IB) = \frac{\pi - \angle A}{2} = \angle(FY, FB), \end{aligned}$$

so  $Q' \in \Omega_B$ . Similarly,  $Q' \in \Omega_C$ . Thus  $Q' \in \Omega_B \cap \Omega_C = \{P, Q\}$  and it remains to prove that  $Q' \neq P$ . If we had  $Q' = P$ , we would have  $\angle(PY, PZ) = \angle(Q'Y, Q'Z) = \angle(IC, IB)$ . This would imply

$$\angle(PY, YF) + \angle(EZ, ZP) = \angle(PY, PZ) = \angle(IC, IB) = \angle(PE, PF),$$

so circles  $\Omega_B$  and  $\Omega_C$  would be tangent at  $P$ . That is excluded in the problem conditions, so  $Q' = Q$ .



*Step 4.* Now we are ready to show that  $P$ ,  $Q$ , and  $S$  are collinear.

Notice that  $A$  and  $D$  are the poles of  $EW$  and  $DW$  with respect to  $\omega$ , so  $W$  is the pole of  $AD$ . Hence,  $WI \perp AD$ . Since  $CI \perp DE$ , this yields  $\angle(IC, WI) = \angle(DE, DA)$ . On the other hand,  $DA$  is a symmedian in  $\triangle DEF$ , so  $\angle(DE, DA) = \angle(DN, DF) = \angle(DS, DF)$ . Therefore,

$$\begin{aligned} \angle(PS, PF) &= \angle(DS, DF) = \angle(DE, DA) = \angle(IC, IW) \\ &= \angle(YC, YW) = \angle(YQ, YF) = \angle(PQ, PF), \end{aligned}$$

which yields the desired collinearity.