Version: English

34th Nordic Mathematical Contest

Solutions

Problems

1. For a positive integer n, denote by $g(n)$ the number of strictly ascending triples chosen from the set $\{1, 2, \dots, n\}$. Find the least positive integer n such that the following holds: The number $g(n)$ can be written as the product of three different prime numbers which form an arithmetic progression with common difference 336.

2. Georg has $2n + 1$ cards with one number written on each card. On one card the integer 0 is written, and among the rest of the cards, the integers $k = 1, \dots, n$ appear, each twice. Georg wants to place the cards in a row in such a way that the 0-card is in the middle, and for each $k = 1, \dots, n$, the two cards with the number k have the distance k (meaning that there are exactly $k - 1$ cards between them).

   For which $1 \leq n \leq 10$ is this possible?

3. Each of the sides AB and CD of a convex quadrilateral $ABCD$ is divided into three equal parts, $|AE| = |EF| = |FB|$, $|DP| = |PQ| = |QC|$. The diagonals of $AEPD$ and $FBCQ$ intersect at M and N, respectively. Prove that the sum of the areas of $\triangle AMD$ and $\triangle BNC$ is equal to the sum of the areas of $\triangle EPM$ and $\triangle FNQ$.

4. Find all functions $f\colon \mathbb{R} \setminus \{-1\} \to \mathbb{R}$ such that

$$f(x)f\left(f\left(\frac{1-y}{1+y}\right)\right) = f\left(\frac{x+y}{xy+1}\right)$$

   for all $x, y \in \mathbb{R}$ that satisfy $(x + 1)(y + 1)(xy + 1) \neq 0$.

Solutions

These are mainly written as provided by proposers.

Problem 1

Antalet sådana talföljder är $\binom{n}{3}$, eftersom följden bestäms entydigt av de tre tal som ingår i den.

Om $\binom{n}{3}$ är produkten av tre skilda primtal måste n vara udda (annars innehåller $n, n-1, n-2$ för många faktorer 2), och vi har tre olika möjligheter (där p_i är olika primtal):

Fall 1: $n = p_1$, $(n - 1) = 6p_2$ och $(n - 2) = p_3$.

Fall 2a: $n = 3p_1$, $(n - 1) = 2p_2$ och $(n - 2) = p_3$.

Fall 2b: $n = p_1$, $(n-1) = 2p_2$ och $(n-2) = 3p_3$.

I fall 1 är $p_1 - p_3 = 2$, så dessa tal kan omöjligtvis ingå i en aritmetisk talföljd med steglängd 336.

I fall 2a får vi $p_1 = \frac{n}{3}$, $p_2 = \frac{n-1}{2}$ och $p_3 = n-2$. Den minsta differensen här är $p_2 - p_1 = \frac{n-1}{2} - \frac{n}{3} = \frac{n-3}{6}$ och som optimister så hoppas vi på att detta steg är $1 \cdot 336$ och inte mer (vilket är teoretiskt minimum). Vi får då $6 \cdot 336 = n - 3$, vilket ger $n = 2019$, och det visar sig då att steget vidare till $n-2$ är $4 \cdot 336$. Här kan man misstänka att detta är det rätta svaret, och det är därför mödan värt att kolla om $2019/3 = 673$, $2018/2 = 1009$ och 2017 är primtal, vilket medför lite jobb, men inte så farligt mycket. Värst är att kolla 2017, vilket innebär koll mot primtalen $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43$.

Det kvarstår att kontrollera om fall 2b kan ge ett mindre n. Här är den minsta differensen $p_2 - p_3 = \frac{n-1}{2} - \frac{n-2}{3} = \frac{n+1}{6}$, och vi hoppas återigen på att detta är $1 \cdot 336$, vilket ger $6 \cdot 336 = n+1$ vilket ger $n = 2015$, men det är ju delbart med 5, vilket utesluter att $\binom{2015}{3}$ kan skrivas som en produkt av exakt tre primtal.

Problem 2

Let the positions be enumerated from $-n$ to n, and let a_k and $a_k - k$ be the positions of the two cards with the number k. By symmetry around 0, we get that

$$0 = \sum_{k=1}^{n} (a_k + a_k - k) = 2 \sum_{k=1}^{n} a_k - \frac{n(n+1)}{2}.$$

It follows that either n or $n+1$ is divisible by 4, id est

$$n \in \{3, 4, 7, 8\}.$$

The following shows that each value is also possible

2 3 2 0 3 1 1
2 4 2 3 0 4 3 1 1
5 3 4 7 3 5 4 0 6 2 7 2 1 1 6
6 8 5 7 1 1 6 5 0 8 7 4 2 3 2 4 3

Problem 3

Denote by $T(XYZ)$ the surface area of $\triangle XYZ$ (similarly for quadrilaterals). First we note the identical areas of triangles

$$\begin{aligned} T(EFP) &= T(AEP), \\ T(PFQ) &= T(QFC), \\ T(EFQ) &= T(FBQ) \text{ and} \\ T(EQP) &= T(EPD) \end{aligned} \tag{1}$$

The sum of the first two line gives

$$T(EFQP) = T(AEP) + T(QFC)$$

and the third and fourth line give

$$T(EFQP) = T(FBQ) + T(EDP). \tag{2}$$

The sum of the last two lines can be written in a way that is close to the statement of the problem

$$2T(EFQP) = T(AEPD) + T(EPM) - T(AMD) + T(FBCQ) + T(FNQ) - T(BNC).$$

This means the problem is equivalent to showing

$$2T(EFQP) = T(AEPD) + T(FBCQ) \tag{3}$$

[If student gets this far, the solution is worth four points.]

The area of quadrilateral is half the magnitude of the cross product of its diagonals. Letting $\bar{t} = \overrightarrow{EA} + \overrightarrow{DP}$ we can write for the three cross products

$$\begin{aligned}
2T(AEPD) &= \overrightarrow{AP} \times \overrightarrow{ED}, \\
2T(EFQP) &= \overrightarrow{EQ} \times \overrightarrow{FP} = (\overrightarrow{AP} + \bar{t}) \times (\overrightarrow{ED} + t) = \overrightarrow{AP} \times \overrightarrow{FP} + \bar{t} \times \overrightarrow{ED} + \overrightarrow{AP} \times \bar{t}, \\
2T(FBCQ) &= \overrightarrow{FC} \times \overrightarrow{BQ} = (\overrightarrow{AP} + 2t) \times (\overrightarrow{ED} \ + 2t) = \overrightarrow{AP} \times \overrightarrow{ED} + 2\bar{t} \times \overrightarrow{ED} + 2\overrightarrow{AP} \times \bar{t}
\end{aligned}$$

from which the result follows.

Problem 4

Setting $y = 1$ we obtain

$$f(x)f\left(f\left(0\right)\right) = f(1)$$

If $f(f(0)) \neq 0$ we can divide and obtain that $f(x) = c$ for some $c \in \mathbb{R}$. The only such solutions are $f(x) = 0$ and $f(x) = 1$. So we can assume $f(f(0)) = f(1) = 0$.

Setting $x = y = 0$ we see that $f(0)f(f(1)) = f(0)^2 = f(0) \Rightarrow f(0) = 0$ or 1.

If $f(0) = 0$ we can set $x = 0$ and get $f(y) = f(0)f\left(f\left(\frac{1-y}{1+y}\right)\right) = 0$.

Therefore we can assume $f(0) = 1$ and get

$$f\left(f\left(\frac{1-y}{1+y}\right)\right) = f\left(y\right) \tag{4}$$

Substituting this back into the original equation we obtain

$$f(x)f(y) = f\left(\frac{x+y}{xy+1}\right) \tag{5}$$

Substituting $z = \dfrac{1-y}{1+y}$ into (4)

$$f\left(f\left(z\right)\right) = f\left(\frac{1-z}{1+z}\right) \tag{6}$$

We will now prove the lemma that $f(x) = 0 \Leftrightarrow x = 1$. We know $f(1) = 0$, so assume $f(a) = 0$ for some $a \neq 1$. Then by (5),

$$f\left(\frac{x+a}{xa+1}\right) = f(x)f(a) = 0 \quad \forall x \in \mathbb{R} \setminus \{-1, -\frac{1}{a}\}$$

If $z = \dfrac{x+a}{xa+1}$, then $x = \dfrac{a-z}{az-1}$. So z can take any value in $\mathbb{R} \setminus \{-1, \frac{1}{a}\}$ because

$$x = -1 \Leftrightarrow 1 - az = a - z \Leftrightarrow (a-1)(z+1) = 0 \Leftrightarrow 1 = 0$$

By setting $x = y = \frac{1}{a}$ in (5) we obtain

$$f\left(\frac{1}{a}\right)^2 = f\left(\frac{2a}{a^2+1}\right) = f\left(\frac{2}{a+\frac{1}{a}}\right) = f(a)^2 = 0$$

So $f\left(\frac{1}{a}\right) = 0$ which means that $f(z) = 0 \ \ \forall z \in \mathbb{R} \setminus \{-1\}$, which contradicts our assumptions.

Now we will show that $f(a) = f(b) \Rightarrow a = b \vee a = \frac{1}{b}$. Assume that $a \neq b$ and $f(a) = f(b)$. We know that $f(b) \neq 0$ by the lemma. By setting $x = -y = b$ in (5) we get

$$f(b)f(-b) = f(0) = 1 \Leftrightarrow \frac{1}{f(b)} = f(-b).$$

Therefore if $a \neq \frac{1}{b}$, we can use (5) and (6) to get

$$0 = f(1) = f\left(\frac{f(a)}{f(b)}\right) = f\left(f(a)f(-b)\right) = f\left(f\left(\frac{a-b}{1-ab}\right)\right) = f\left(\frac{1 - \dfrac{a-b}{1-ab}}{1 + \dfrac{a-b}{1-ab}}\right).$$

So by the lemma:

$$\frac{1 - \dfrac{a-b}{1-ab}}{1 + \dfrac{a-b}{1-ab}} = 1 \quad \Rightarrow \quad (1-ab) - a + b = (1-ab) + a - b \quad \Rightarrow \quad a = b$$

Combining this result with (6), we see that for each $x \in \mathbb{R} \setminus \{-1\}$ we have

$$f(x) = \frac{1-x}{1+x} \quad \vee \quad f(x) = \frac{1+x}{1-x}$$

Suppose that $\exists c \notin \{1, 0, -1\}$ such that $d = f(c) = \dfrac{1+c}{1-c}$. Then

$$f(d) = f(f(c)) = f\left(\frac{1-c}{1+c}\right) = f\left(\frac{1}{d}\right) \quad \Rightarrow$$

$$f(x)f(d) = f\left(\frac{x+d}{xd+1}\right) = f(x)f\left(\frac{1}{d}\right) = f\left(\frac{x+\frac{1}{d}}{\frac{x}{d}+1}\right) = f\left(\frac{xd+1}{x+d}\right)$$

As before, the substitution $z = \dfrac{x+d}{xd+1}$ ranges over all of $\mathbb{R} \setminus \{-1, \frac{1}{d}\}$, giving us that $f(z) = f(\frac{1}{z})$ for all $z \in \mathbb{R} \setminus \{-1\}$. This means in particular that $f(\frac{1}{c}) = f(c) = \dfrac{1+c}{1-c}$. But this value is neither $\dfrac{1+\frac{1}{c}}{1-\frac{1}{c}}$ nor $\dfrac{1-\frac{1}{c}}{1+\frac{1}{c}}$, leading to a contradiction. Therefore we must have that $f(x) = \dfrac{1-x}{1+x}$ for all $x \in \mathbb{R} \setminus \{-1\}$, which indeed is our final solution.