Finnish National High School Mathematics Competition

Final Rounds 1997–2008

The Finnish National High School Mathematics Competition is organized by MAOL, the The Finnish Association of Teachers of Mathematics, Physics, Chemistry and Informatics. Since 1997, the competion has been arranged in two stages: In the first round, which is held in three age categories, the contestants of the final are selected. This year the quotas were 15 representatives from the oldest age category, 4 from the second highest and 1 from the youngest category. There is only one problem set in the final, but the students are awarded prizes according to their category. Below are the translations of the problem sets of all of the finals.

25 January 1997

1. Determine the real numbers a such that the equation

$$a \cdot 3^x + 3^{-x} = 3$$

has exactly one solution x.

2. Circles with radii R ja r $(R > r)$ are externally tangent. Another common tangent of the circles in drawn. This tangent and the circles bound a region inside which a circle as large as possible is drawn. What is the radius of this circle?

3. 12 knights are sitting at a round table. Every knight is an enemy with two of the adjacent knights but with none of the others. 5 knights are to be chosen to save the princess, with no enemies in the group. How many ways are there for the choice?

4. Count the sum of the four-digit positive integeres containing only odd digits in their decimal representation.

5. For an integer $n \geq 3$, place n points on the plane in such a way that all the distances between the points are at most one and exactly n of the pairs of points have the distance one.

30 January 1998

1. Show that points A, B, C and D can be placed on the plane in such a way that the quadrilateral $ABCD$ has an area which is twice the area of the quadrilateral $ADBC$.

2. There are 11 members in the competetion committee. The problem set is kept in a safe having several locks. The committee members have been provided with keys in such a way that every six members can open the safe, but no five members can do that. What is the smallest possible number of locks, and how many keys are needed in that case?

3. Consider the geometric sequence $1/2$, $1/4$, $1/8, \ldots$ Can one choose a subsequence, finite or infinite, for which the ratio of consecutive terms is not 1 and whose sum is $1/5$?

4. There are 110 points in a unit square. Show that some four of these points reside in a circle whose radius is $1/8$.

5. 15×36-checkerboard is covered with square tiles. There are two kinds of tiles, with side 7 or 5. Tiles are supposed to cover whole squares of the board and be non-overlapping. What is the maximum number of squares to be covered?

22 January 1999

1. Show that the equation

$$x^3 + 2y^2 + 4z = n$$

has an integral solution (x, y, z) for all integers n.

2. Suppose that the positive numbers $a_1, a_2, \ldots, a_n$ form an arithmetic progression; hence $a_{k+1} - a_k = d$, for $k = 1, 2, \ldots, n-1$. Prove that

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \cdots + \frac{1}{a_{n-1} a_n} = \frac{n-1}{a_1 a_n}.$$

3. Determine how many primes are there in the sequence

$$101,\ 10101,\ 1010101 \ldots$$

4. Three unit circles have a common point O. The other points of (pairwise) intersection are A, B and C. Show that the points A, B and C are located on some unit circle.

5. An ordinary domino tile can be identifies as a pair (k, m) where numbers k and m can get values 0, 1, 2, 3, 4, 5 and 6. Pairs (k, m) and (m, k) determine the same tile. In particular, the pair (k, k) determines one tile. We say that two domino tiles *match*, if they have a common component. *Generalized n-domino tiles* m and k can get values $0, 1, \ldots, n$. What is the probability that two randomly chosen n-domino tiles match?

28 January 2000

1. Two circles are externally tangent at the point A. A common tangent of the circles meets one circle at the point B and another at the point C ($B \neq C$). Line segments BD and CE are diameters of the circles. Prove that the points D, A and C are collinear.

2. Prove that the integral part of the decimal representation of the number $(3 + \sqrt{5})^n$ is odd, for every positive integer n.

3. Determine the positive integers n such that the inequality

$$n! > \sqrt{n^n}$$

holds.

4. There are seven points on the plane, no three of which are collinear. Every pair of points is connected with a line segment, each of which is either blue or red. Prove that there are at least four monochromatic triangles in the figure.

5. Irja and Valtteri are tossing coins. They take turns, Irja starting. Each of them has a pebble which reside on opposite vertices of a square at the start. If a player gets heads, she or he moves her or his pebble on opposite vertex. Otherwise the player in turn moves her or his pebble to an adjacent vertex so that Irja proceeds in positive and Valtteri in negative direction. The winner is the one who can move his pebble to the vertex where opponent's pebble lies. What is the probability that Irja wins the game?

2 February 2001

1. In the right triangle ABC, CF is the altitude based on the hypotenuse AB. The circle centered at B and passing through F and the circle with centre A and the same radius intersect at a point of CB. Determine the ratio $FB : BC$.

2. Equations of non-intersecting curves are $y = ax^2 + bx + c$ and $y = dx^2 + ex + f$ where $ad < 0$. Prove that there is a line of the plane which does not meet either of the curves.

3. Numbers a, b and c are positive integers and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$. Show that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{41}{42}.$$

4. A sequence of seven digits is randomly chosen in a weekly lottery. Every digit can be any of the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$. What is the probability of having at most five different digits in the sequence?

5. Determine $n \in \mathbb{N}$ such that $n^2 + 2$ divides $2 + 2001n$.

8 February 2002

1. A function f satisfies $f(\cos x) = \cos(17x)$, for every real x. Show that $f(\sin x) = \sin(17x)$, for every $x \in \mathbb{R}$.

2. Show that if

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c},$$

then also

$$\frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n} = \frac{1}{a^n + b^n + c^n},$$

provided n is an odd positive integer.

3. n pairs are formed from n girls and n boys at random. What is the probability of having at least one pair of girls? For which n the probability is over 0.9?

4. Convex figure $\mathcal{K}$ has the following property: if one looks at $\mathcal{K}$ from any point of the certain circle $\mathcal{Y}$, then $\mathcal{K}$ is seen in the right angle. Show that the figure is symmetric with respect to the centre of $\mathcal{Y}$.

5. There is a regular 17-gon $\mathcal{P}$ and its circumcircle $\mathcal{Y}$ on the plane. The vertices of $\mathcal{P}$ are coloured in such a way that $A, B \in \mathcal{P}$ are of different colour, if the shorter arc connecting A and B on $\mathcal{Y}$ has $2^k + 1$ vertices, for some $k \in \mathbb{N}$, including A and B. What is the least number of colours which suffices?

7 February 2003

1. The incentre of the triangle ABC is I. The rays AI, BI and CI intersect the circumcircle of the triangle ABC at the points D, E and F, respectively. Prove thatAD and EF are perpendicular.

2. Find consecutive integers bounding the expression

$$\frac{1}{x_1+1} + \frac{1}{x_2+1} + \frac{1}{x_3+1} + \cdots + \frac{1}{x_{2001}+1} + \frac{1}{x_{2002}+1}$$

where $x_1 = 1/3$ and $x_{n+1} = {x_n}^2 + x_n$.

3. There are six empty purses on the table. How many ways are there to put 12 two-euro coins in purses in such a way that at most one purse remains empty?

4. Find pairs of positive integers (n, k) satisfying

$$(n+1)^k - 1 = n!.$$

5. Players Aino and Eino take turns choosing numbers from the set $\{0, \ldots, n\}$ with $n \in \mathbb{N}$ being fixed in advance. The game ends when the numbers picked by one of the players include an arithmetic progression of length 4. The one who obtains the progression wins. Prove that for some n, the starter of the game wins. Find the smallest such n.

6 February 2004

1. The equations $x^2 + 2ax + b^2 = 0$ and $x^2 + 2bx + c^2 = 0$ both have two different real roots. Determine the number of real roots of the equation $x^2 + 2cx + a^2 = 0$.

2. a, b ja c are positive integers and

$$\frac{a\sqrt{3}+b}{b\sqrt{3}+c}$$

is a rational number. Show that

$$\frac{a^2+b^2+c^2}{a+b+c}$$

is an integer.

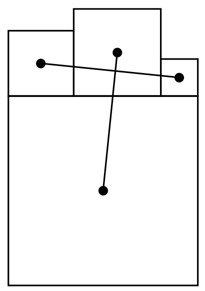
3. Two circles with radii r and R are externally tangent. Determine the length of the segment cut from the common tangent of the circles by the other common tangents.

4. The numbers $2005! + 2$, $2005! + 3$, $\ldots$, $2005! + 2005$ form a sequence of 2004 consequtive integers, none of which is a prime number. Does there exist a sequence of 2004 consequtive integers containing exactly 12 prime numbers?

5. Finland is going to change the monetary system again and replace the Euro by the Finnish Mark. The Mark is divided into 100 pennies. There shall be coins of three denominations only, and the number of coins a person has to carry in order to be able to pay for any purchase less than one mark should be minimal. Determine the coin denominations.

4 February 2005

1. In the figure below, the centres of four squares have been connected by two line segments. Prove that these line segments are perpendicular.



2. There are 12 seats at a round table in a restaurant. A group of five women and seven men arrives at the table. How many ways a there for choosing the sitting order, provided that every woman ought to be surrounded by two men and two orders are regarded as different, if at least one person has a different neighbour on one's right side.

3. Solve the group of equations

$$\begin{cases} (x+y)^3 = z \\ (y+z)^3 = x \\ (z+x)^3 = y. \end{cases}$$

4. The numbers 1, 3, 7 and 9 occur in the decimal representation of an integer. Show that permuting the order of digits one can obtain an integer divisible by 7.

5. A finite sequence is said to be *disorderly*, if no two terms of the sequence have their average in between them. For example, $(0, 2, 1)$ is disorderly, for $1 = \frac{0+2}{2}$ is not in between 0 and 2, and the other averages $\frac{0+1}{2} = \frac{1}{2}$ and $\frac{2+1}{2} = 1\frac{1}{2}$ do not even occur in the sequence. Prove that for every $n \in \mathbb{N}$ there is a disorderly sequence enumerating the numbers $0, 1, \ldots, n$ without repertitions.

3 February 2006

1. Determine all pairs (x, y) of positive integers for which the equation

$$x + y + xy = 2006$$

holds.

2. Show that the inequality

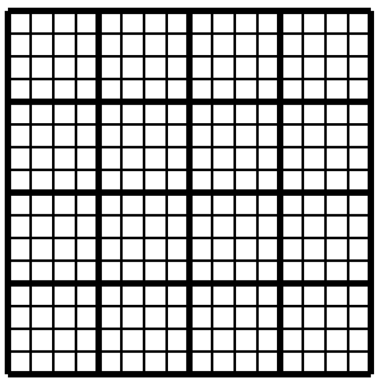
$$3(1 + a^2 + a^4) \geq (1 + a + a^2)^2$$

holds for all real numbers a.

3. The numbers p, $4p^2 + 1$, and $6p^2 + 1$ are primes. Determine p.

4. Two medians of a triangle are perpendicular. Prove that the medians of the triangle are the sides of a right-angled triangle.

5. The game of *Nelipe* is played on a 16×16-grid as follows: The two players write in turn numbers $1, 2, \ldots, 16$ in different squares. The numbers on each row, column, and in every one of the 16 smaller squares have to be different. The loser is the one who is not able to write a number. Which one of the players wins, if both play with an optimal strategy?



2 February 2007

1. Show: when a prime number is divided by 30, the remainder is either 1 or a prime number. Is a similar claim true, when the divisor is 60 or 90?

2. Determine the number of real roots of the equation

$$x^8 - x^7 + 2x^6 - 2x^5 + 3x^4 - 3x^3 + 4x^2 - 4x + \frac{5}{2} = 0.$$

3. There are five points in the plane, no three of which are collinear. Show that some four of these points are the vertices of a convex quadrilateral.

4. The six offices of the city of Salavaara are to be connected to each other by a communication network which utilizes modern picotechnology. Each of the offices is to be connected to all the other ones by direct cable connections. Three operators compete to build the connections, and there is a separate competition for every connection. When the network is finished one notices that the worst has happened: the systems of the three operators are incompatible. So the city must reject connections built by two of the operators, and these are to be chosen so that the damage is minimized. What is the minimal number of offices which still can be connected to each other, possibly through intermediate offices, in the worst possible case.

5. Show that there exists a polynomial $P(x)$ with integer coefficients such that the equation $P(x) = 0$ has no integer solutions but for each positive integer n there is an $x \in \mathbb{Z}$ such that $n \mid P(x)$.

1 February 2008

1. Foxes, wolves and bears arranged a big rabbit hunt. There were 45 hunters catching 2008 rabbits. Every fox caught 59 rabbits, every wolf 41 rabbits and every bear 40 rabbits. How many foxes, wolves and bears were there in the hunting company?

2. The incentre of the triangle ABC is I. The lines AI, BI and CI meet the circumcircle of the triangle ABC also at points D, E and F, respectively. Prove that AD and EF are perpendicular.

3. Solve the Diofantine equation

$$x^{2008} - y^{2008} = 2^{2009}.$$

4. Eight football teams play matches against each other in such a way that no two teams meet twice and no three teams play all of the three possible matches. What is the largest possible number of matches?

5. The closed line segment I is covered by finitely many closed line segments. Show that one can choose a subfamily S of the family of line segments having the properties:
(1) the chosen line segments are disjoint,
(2) the sum of the lengths of the line segments of S is more than half of the length of I.

Show that the claim does not hold any more if the line segment I is replaced by a circle and other occurences of the compound word “line segment” by the word “circular arc”.