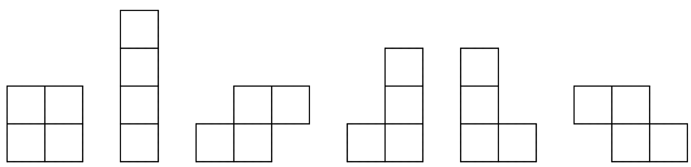
SEVENTH GRADERS' FINAL ROUND IN HELSINKI 3.3.2012 SOLUTIONS

(1) Compute
$$\frac{-2012}{1+(-2012)^2}+\ldots+\frac{-1}{1+(-1)^2}+\frac{0}{1+0^2}+\frac{1}{1+1^2}+\ldots+\frac{2012}{1+2012^2}.$$
Solution. It turns out that the sum of the first and the last terms is zero, the sum of the second and the second to the last terms is zero, and so on. More precisely, when $n = 1, 2, \ldots, 2012$, we have
$$\frac{-n}{1+(-n)^2}+\frac{n}{1+n^2}=\frac{-n}{1+n^2}+\frac{n}{1+n^2}=0.$$
Thus the value of the expression is zero.

(2) Find numbers a, b and c for which the equation
$$x^3 + ax^2 + bx + c = 0$$
has the solutions $x = 3$, $x = 4$ and $x = 5$. (Here $x^3 = x \cdot x \cdot x$ and $x^2 = x \cdot x$.)

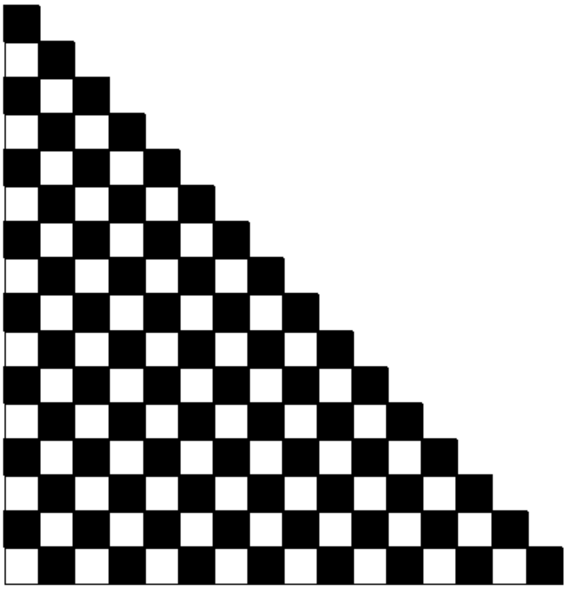
Solution. The equation
$$(x-3)(x-4)(x-5) = 0$$
has the solutions $x = 3$, $x = 4$ and $x = 5$. By expanding the left-hand side the equation takes the form
$$x^3 - (3+4+5)x^2 + (3\cdot 4 + 3\cdot 5 + 4 \cdot 5)x - 3\cdot 4\cdot 5 = 0$$
which simplifies to
$$x^3 - 12x^2 + 47x - 60 = 0.$$

(3) We have the following kinds of Tetris tiles at our disposal, an unlimited supply of each:



These may be rotated freely every which way. Is it possible to cover the following figure using these tiles in such a manner that no two tiles overlap and no tile extends beyond the boundary of the figure?

Solution. Let us color the squares of the figure with black and white as one usually colors the squares of a chessboard, and let us do it so that the squares in the longest diagonal are black. Now each of our Tetris tiles covers precisely two black and two white squares. But our figure has more black than white squares! Perhaps the easiest way to see this is to observe that every other row has equally many black as it has white squares, and every other row has precisely one black square more than it has white squares.



(4) What is the time in the precision of one second, when it is between one o'clock and two o'clock and the minute and hour hands are pointing exactly to the same direction?

Solution. Let the time be x minutes past one (where x need not be a whole number). The angle between the minute hand and a vertical line is

$$\frac{x}{60} \cdot 360° = x \cdot 6°.$$

The hour hand rotates an angle of $\frac{360°}{12}$ in one hour, so that during the moment we are considering, its angle with a vertical line is

$$\frac{360°}{12} + \frac{x}{60} \cdot \frac{360°}{12}.$$

Since the hands point to the same direction, we then must have

$$\frac{360°}{12} + \frac{x}{60} \cdot \frac{360°}{12} = x \cdot 6°.$$

By multiplying both sides by 12 and dividing both sides by 6°, we obtain

$$60 + x = 12x.$$

In particular, $11x = 60$ and

$$x = \frac{60}{11} = 5{,}45454545...$$

The correct number of seconds has to be

$$60 \cdot \left(\frac{60}{11} - 5\right) = 60 \cdot \frac{5}{11} = \frac{300}{11} = 27{,}2...$$

The time we were looking for is 1:05:27.

(5) Which of the products $1 \cdot 2011$, $2 \cdot 2010$, $3 \cdot 2009$, $\ldots$, $2010 \cdot 2$ and $2011 \cdot 1$ (all products of two positive integers, for which the sum of the factors is 2012) is the largest?

Solution. Let us consider the product $(1006 - n)(1006 + n)$, where n is an integer for which $-1006 < n < 1006$. Now

$$(1006 - n)(1006 + n) = 1006^2 - n^2.$$

Since we always have $n^2 \geqslant 0$, the product must be largest when $n = 0$. Thus the largest of the products is the middle one, i.e. $1006 \cdot 1006$.