Mathematics Competition for the Seventh Graders of Helsinki 2013/2/7
Solutions

1. A bookstore sells pocket books nine euros apiece, and they have started a marketing campaign during which after buying seven pocket books one gets an eight for free. A friend of books finds 56 interesting pocket books in the bookstore. How much do they cost?

a) 392 € **b)** 432 € **c)** 441 € **d)** 495 € **e)** 504 €

Solution. Each eight books cost the same as seven, which is $7 \cdot 9$€=63€, and the eight is free. Because we are considering $56 = 7 \cdot 8$ books, they must cost $7 \cdot 63$€ $= 441$€.

2. The measures of the floor of a room are 3×5 metres, and its height is 3 metres. Tiling the floor requires 60 tiles. How many tiles are needed to tile the entire room (including the walls, the floor and the ceiling)?

a) 240 **b)** 312 **c)** 360 **d)** 372 **e)** 390

Solution. Since the area of the floor is $3 \cdot 5 = 15$ square meters, and tiling it requires 60 tiles, one square meter must require $\frac{60}{15} = 4$ tiles. The areas of either of the long walls of the of the room is $15\,\mathrm{m}^2$, the area of the ceiling is the same as that of the floor, i.e. $15\,\mathrm{m}^2$, and the area of either of the short walls is $3 \cdot 3 = 9$ square meters. Therefore the combined area of the inner surfaces of the room (floor, long walls, ceiling, short walls) is

$$15 + 2 \cdot 15 + 15 + 2 \cdot 9 = 78\,\text{square meters.}$$

Therefore, $4 \cdot 78 = 312$ tiles are needed.

3. In a certain country, one inhabitant consumes 12 kg of coffee per year in average. How many tons of coffee is consumed in the country per year, if its population is 5,3 million people?

a) less than 10 tons
b) more than 10 and less than 100 tons
c) more than 100 and less than 1000 tons
d) more than 1000 and less than 10000 tons
e) more than 10000 tons

Solution. One thousand inhabitants consume 12 tons of coffee per year. The consumption in the entire country is 5300 times this, which is $12 \cdot 5300 = 63600$ tons. This is clearly more than 10000 tons.

4. What is

$$\underbrace{2012 + 2012 + 2012 + \ldots + 2012}_{\text{2012 appears here 2013 times}} - \underbrace{2013 - 2013 - 2013 - \ldots - 2013}_{\text{2013 appears here 2012 times}}\ ?$$

a) -4025 **b)** -2013 **c)** 0 **d)** 1 **e)** 2012

Solution. The number is $2013 \cdot 2012 - 2012 \cdot 2013 = 0$.

5. A small child has four building blocks of different colors, and has firmly decided to build from them a tower four blocks high. In how many different orders can colors appear in the tower?

a) 10 **b)** 12 **c)** 16 **d)** 18 **e)** 24

Solution. Suppose the tower is build from ground up. The color of the first block may be chosen in four different ways. The color of the next block may be chosen in three different ways. For the third block, there will be only two colors left, and the color of the last block is uniquely determined by those of the first three. Thus, the total number of different orders the colors may appear in is $4 \cdot 3 \cdot 2 \cdot 1 = 24$.

If the colors are a, b, c and d, then it is possible to even list all the possibilities:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
d	c	d	b	c	b	d	c	d	a	c	a	d	b	d	a	b	a	c	b	c	a	b	a
c	d	b	d	b	c	c	d	a	d	a	c	b	d	a	d	a	b	b	c	a	c	a	b
b	b	c	c	d	d	a	a	c	c	d	d	a	a	b	b	d	d	a	a	b	b	c	c
a	a	a	a	a	a	b	b	b	b	b	b	c	c	c	c	c	c	d	d	d	d	d	d

6. It is known about two integers x and y, that they are both odd and that their difference is seven. How many such pairs of numbers are there?

a) none **b)** one **c)** five **d)** more than a hundred, but less than a thousand
e) infinitely many

Solution. The difference of two odd numbers is necessarily even, but the number seven is odd. Therefore there are no such pairs of integers.

7. Maija is traveling by train. The train is moving at the speed of 180 km/h when she sees a water tower in front from an angle of 45° from the railway track. In order to find out how far away the tower is from the track, she decides to start the timer of her cellphone, and after ten seconds she sees the tower at behind, again from a 45° angle from the track. How far away is the tower approximately from the track?

a) 250 m **b)** 350 m **c)** 500 m **d)** 700 m
e) the given data are not enough to answer the question

Solution. During those ten seconds the train travels

$$10\,\mathrm{s} \cdot 180\frac{\mathrm{km}}{\mathrm{h}} = 10\,\mathrm{s} \cdot \frac{180\,\mathrm{km}}{3600\,\mathrm{s}} = \frac{180}{360}\,\mathrm{km} = \frac{1}{2}\,\mathrm{km} = 500\,\mathrm{m}.$$

The position of the train in the beginning, its position in the end, and the water tower form an isosceles triangle whose vertex is at the water tower. The base of the triangle is the distance the train traveled during the ten seconds. The distance between the water tower and the railway track is the height of the triangle, which is half of the length of its base: $\frac{1}{2} \cdot 500\,\mathrm{m} = 250\,\mathrm{m}$.

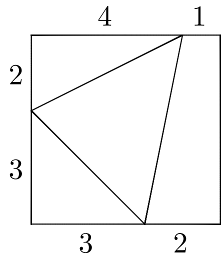
8. Of five numbers the first is 1 and the last is 9. Furthermore, the product of any three consecutive numbers in the list is 3. What is the number in the middle of the list?

a) $\frac{1}{3}$ **b)** $\frac{1}{\sqrt{3}}$ **c)** 1 **d)** $\sqrt{3}$ **e)** 3

Solution. Let the numbers be a, b, c, d and e. Because none of the products of three consecutive numbers vanishes, none of the numbers can be zero. Since $bcd = cde$, we must have $b = e = 9$. Since

$$3 = abc = 1 \cdot 9 \cdot c, \quad \text{we must have} \quad c = \frac{3}{9} = \frac{1}{3}.$$

9. A triangle is drawn inside a square so that the vertices of the triangle are on the sides of the square



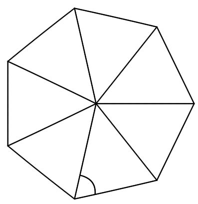
What is the area of the triangle?

a) 14 **b)** 15 **c)** 16 **d)** less than 14 **e)** more than 16

Solution. The area of the square is $5 \cdot 5 = 25$. The area of the triangle at the upper left corner is $\frac{1}{2} \cdot 2 \cdot 4 = 4$. The area of the triangle at the bottom left corner is $\frac{1}{2} \cdot 3 \cdot 3 = \frac{9}{2}$. The part of the square on the right of the center triangle can be partitioned into a 1×5-rectangle, whose area is 5, and a triangle with base 1 and height 5. The area of the last triangle is $\frac{1}{2} \cdot 1 \cdot 5 = \frac{5}{2}$. The area of the triangle in the center is what remains of the area of the square when the areas of everything else in it is subtracted from it:

$$25 - 4 - \frac{9}{2} - 5 - \frac{5}{2} = 16 - \frac{14}{2} = 16 - 7 = 9.$$

10. In the following picture there is a regular 7-gon (i.e. a "heptagon"), and its vertices have been joined to its center using straight line segments. How large is the angle marked into the picture?



a) $50°$ **b)** $51\frac{3}{7}°$ **c)** $60°$ **d)** $64\frac{2}{7}°$ **e)** $72\frac{4}{7}°$

Solution. There are seven identical isosceles triangles in the picture. The sum of their vertex angles is $360°$, and so each vertex angle is $\frac{360}{7}°$. The sum of the angles of a triangle is always $180°$, and therefore the sum of the base angles of a single triangle in the picture is $180° - \frac{360}{7}°$, and one base angle must be half of this:

$$\frac{180° - \frac{360}{7}°}{2} = 90° - \frac{180}{7}° = \frac{630° - 180°}{7} = \frac{450}{7}° = 60° + \frac{30}{7}° = 64\frac{2}{7}°.$$

11. If A is a number such that $A^2 + A + 1 = 0$, then what is $\frac{1}{A^2}$?

a) 1 **b)** A **c)** A^2 **d)** 0 **e)** -1

Solution. Let us begin by observing that

$$0 = (A-1)\left(A^2 + A + 1\right) = A^3 + A^2 + A - A^2 - A - 1 = A^3 - 1,$$

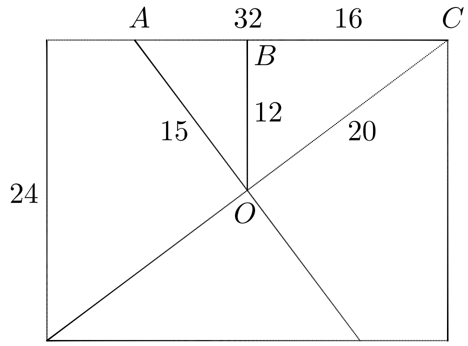
so that $A^3 = 1$. It follows that

$$\frac{1}{A^2} = 1 \cdot \frac{1}{A^2} = A^3 \cdot \frac{1}{A^2} = A.$$

12. The measures of a sheet of paper are 24×32. It is folded so that one corner comes to lie on the opposite corner. How long is the crease (i.e. the line segment created by the folding)?

a) 26 **b)** 28 **c)** 30 **d)** 32 **e)** 34

Solution. The crease is orthogonal to the diagonal corresponding to the corners which come on top of each other, and it passes through the center of the piece of paper. Let A, B, C and O be as in the following figure:



By Pythagoras' theorem, the length of the diagonal is $\sqrt{24^2 + 32^2} = 40$, that is $OC = 20$. Furthermore, $OB = 12$ and $BC = 16$. The triangles $\triangle ABO$ and $\triangle OBC$ are similar, and therefore

$$\frac{AO}{BO} = \frac{OC}{BC}, \quad \text{so that} \quad AO = 12 \cdot \frac{20}{16} = 12 \cdot \frac{5}{4} = 3 \cdot 5 = 15.$$

Now the length of the crease must be $2 \cdot AO = 2 \cdot 15 = 30$.

13. The numbers

$$0^2, \quad 1^2, \quad 2^2, \quad 3^2, \quad 4^2, \quad \ldots, \qquad \text{i.e.} \qquad 0, \quad 1, \quad 4, \quad 9, \quad 16, \quad \ldots,$$

are called square numbers. When a square number is divided by five, what are the possible remainders?

a) 0 and 1 **b)** 0, 1 and 2 **c)** 0, 1 and 4 **d)** 0, 1, 3 and 4 **e)** 0, 1, 2, 3 and 4

Solution. If the last digit of a number is subtracted from the number, the result will be divisible by ten, and therefore also by five. Thus, the remainder of a number under division by five is the same as the remainder of its last digit under division by five. Thus the question reduces to the following: which are the possible last digits of a square number?

The last digit of a product of numbers only depends on the last digits of the factors, and so it suffices to consider only the first ten squares which are

$$0, \quad 1, \quad 4, \quad 9, \quad 16, \quad 25, \quad 36, \quad 49, \quad 64, \quad \text{and} \quad 81.$$

The possible last digits are 0, 1, 4, 5, 6 and 9, and the corresponding possible remainders are 0, 1 and 4.

14. In how many ways can we choose positive integers x and y so that $x^4 + y = 10001$?

a) in one way
b) in 10 different ways
c) in 100 different ways
d) in infinitely many different ways
e) not possible

Solution. In each such pair of integers we have $y = 10001 - x^4$, so that the value of x uniquely determines the value of y. Conversely, if x is a positive integer such that $10001 - x^4$ is positive, then these two numbers form a pair of the desired kind. Therefore there are as many pairs with the desired properties as there positive integers x with $10001 - x^4$ positive. The last condition may also be written as $x^4 < 10001$, which means the same as $x^4 \leqslant 10000$. But $10000 = 10^4$. Thus the possible values of the number x^4 are just 1^4, 2^4, 3^4, $\ldots$, 10^4, and the only possible values of x are 1, 2, 3, $\ldots$, 10, and there are exactly ten of these.