The Final Round of the Mathematics Competition of Seventh Graders of Helsinki on 21 April, 2016

In problems 1, 2 and 3 only the answer needs to be given, and the answer is a positive integer, i.e. one of the numbers 1, 2, 3, 4, . . . Each of the problems 1, 2 and 3 is worth three points.

1. Pascal's triangle is defined by first writing 1, and then under it two ones, and each new row is one number longer than the previous ones, it begins and ends with a one, and the numbers in between are each obtained by adding together the two numbers above it:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{array}$$

How many times does the number 10 appear in Pascal's triangle?

2. What is the one hundreth decimal in the number 3/7?

3. We know that

$$1^3 + 2^3 + 3^3 + 4^3 + \ldots + 10^3 = 3025,$$

and that

$$1^3 + 2^3 + 3^3 + 4^3 + \ldots + 20^3 = 44100.$$

What is

$$1^3 + 3^3 + 5^3 + 7^3 + \ldots + 19^3?$$

In problems 4, 5 and 6 the correct answer is not enough and also justifications and intermediate steps need to be given, and these are very important in the grading of the problems. Each of the problems 4, 5 and 6 is worth six points.

4. Let us consider sequences formed from the symbols $\heartsuit$ and $\diamondsuit$. We are allowed to perform three sorts of operations on our sequences:

- We can always erase two consecutive $\heartsuit$ symbols;
- we can always replace the consecutive symbols $\heartsuit\diamondsuit\heartsuit$ by the consecutive symbols $\diamondsuit\diamondsuit$; and
- we can always replace the consecutive symbols $\diamondsuit\diamondsuit$ by the consecutive symbols $\heartsuit\diamondsuit\heartsuit$.

Determine how these operations can be used to turn the sequence $\diamondsuit\diamondsuit\diamondsuit\diamondsuit\diamondsuit\diamondsuit\diamondsuit$ into the sequence $\diamondsuit$.

5. We know that when a positive integer is divided by five, the remainder is two. What are the possible remainders, when the number in question is divided by seven?

6. There are 100 lines in the plane. Some of them may be parallel with one or more other lines, but no three of them intersect at the same point. Is it possible that the lines have alltogether exactly 2016 points of intersection?