

# High School Math Contest, Final Round, February 3, 2012

1. A chord divides a circle into two segments. A square is inscribed in each of them in such a way that two vertices of the square are on the chord and two on the circumference. The ratio of the sidelengths of the squares is 5 : 9. Compute the ratio of the length of the chord to the radius of the circle.

2. Assume  $x \neq 1$ ,  $y \neq 1$  and  $x \neq y$ . Show that if

$$\frac{yz - x^2}{1 - x} = \frac{zx - y^2}{1 - y},$$

then

$$\frac{yz - x^2}{1 - x} = \frac{zx - y^2}{1 - y} = x + y + z.$$

3. Prove that the number  $k^{k-1} - 1$  is divisible by the number  $(k - 1)^2$ , for every integer  $k \geq 2$ .

4. Let  $k, n \in \mathbb{N}$ ,  $0 < k \leq n$ . Prove that

$$\sum_{j=1}^k \binom{n}{j} = \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k} \leq n^k.$$

5. The *Collatz function* is a mapping  $f : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$  such that

$$f(x) = \begin{cases} 3x + 1 & \text{for odd } x, \\ x/2 & \text{for even } x. \end{cases}$$

We denote  $f^1 = f$  and, inductively,  $f^{k+1} = f \circ f^k$ , i.e.  $f^k(x) = \underbrace{f(\dots(f(x)\dots))}_{k \text{ times}}$ .

Prove that there is an  $x \in \mathbb{Z}_+$  such that

$$f^{40}(x) > 2012x.$$