Finnish High School Mathematics Contest
2011 – 2012

The Finnish High School Mathematics contest is organized annually by the Finnish Association of Mathematcs ans Science Teachers. It is the first stage in the Finnish IMO Team selection process. The contest takes place in two rounds. Round One is divided into three divisions acording to the age of the contestants, the Open Division, however, is open for all students. Round One was organized in schools on November 1, 2011. Time allowed was 120 minutes. Altogether about 1500 students took part, evenly devided between the divisions. Round Two was in Helsinki on February 3, 2012. 20 best students from Round One were invited, most of them from the Open Division. In Round Two, the working time was 180 minutes.

In Round One, Basic and Intermediate Division, part of the problems were multiple choice. The number of correct answers to these was not restricted to one for each problem.

Problems

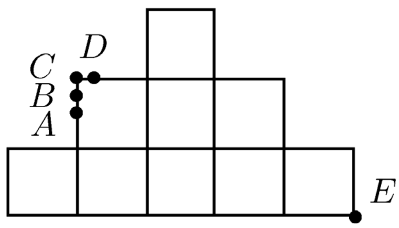
Round One, Basic Division

1. When the number $5^{140} \cdot 8^{47}$ is written in the usual manner, the number of digits is

a) an odd number b) 47 c) 48 d) 141

2. The adjacent figure is composed of nine squares, each of side 1. The line segments AB, BC and CD have length 1/4. One of the lines AE, BE, CE, DE divides the figure into two parts of equal area. Which one?

a) AE b) BE c) CE d) DE



3. Which of the following statements about the diagonals of a hexagon are true?

a) A hexagon has less than ten diagonals.
b) The diagonals of a convex hexagon can have one point in common.
c) A hexagon can have two non-intersecting diagonals.
d) A regular hexagon has two parallel diagonals.

4. The lengths of the sides of a triangle are $2a$, a^2+1 and a^2-1, where $a>1$. The triangle then has the properties:
a) The largest angle can be obtuse. b) The largest angle is a right angle.
c) a^2+1 is the longest side. d) The shortest side depends on the value of a.

5. We know that the function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=ax^5+bx^3+cx+2$ satisfies $f(3)=5$. Then one can conclude that
a) $f(0)=2$ b) $f(-3)=-5$ c) $f(-3)=-1$, d) $f(3)+f(-3)=8$

6. The digits of the fifth power of an integer can be all different from each other as in $2^5=32$ and $3^5=243$, or some digits may repeat as in $10^5=100000$. The number of positive integers whose fifth powers have all their digits different from each other is
a) at least 70 b) at least 90 c) at most 100, d) more than 1000

7. Two equally long trains on adjoining tracks with velocities u and v, $u>v>0$. If the trains travel in the same direction their passing time is twice as long as their passing time, when they travel in opposite directions. (The *passing time* is the time when the trains are at least in part beside each other.) Find the ratio u/v.

8. Prove that the inequality $x^6-x^3+x^2-x+1>0$ holds for all real numbers x.

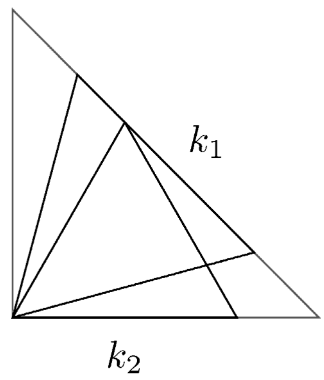
Round One, Intermediate Division

1. Three gamblers playd on money. At the start of the game they had money in ratios $6:5:4$ and at the end they money in ratios $7:6:5$. One of the gamblers won 3 euros. How many euros did he have at the end?
a) 72 b)75 c) 90 d) 108

2. What can be inferred of the solutions of the equation $x^{2011}+x+1=0$?
a) The equation has a unique real solution.
b) The equation has at least one rational solution.
c) The equation has no negative solutions.
d) All solutions of the equation lie in the interval $[-1, 1]$.

3. Which of the following claims concerning the number 211 are true?
a) 211 is a prime number. b) 211 is the product of two prime numbers.
c) 211 is the sum of two prime numbers. d) 211 is the sum of three prime numbers.

4. The legs of an isosceles right triangle are of length a. Inside this triangle there are two equilateral trinagles k_1 and k_2. One vertex of both triangles coincides with the vertex of the right angle. One side of k_1 lies on the hypotenuse and one side of k_2 lies on a leg and one vertex of k_2 is on the hypoptenuse. Determine the ratio of the lengths of the sides of k_1 and k_2.



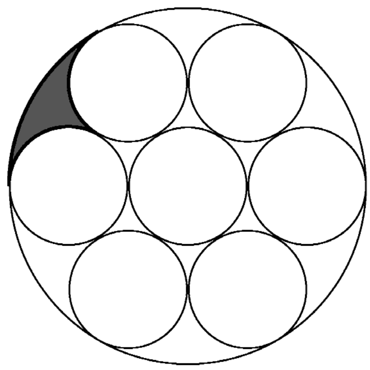
5. Solve the equation

$$(x^2 + y^2 - 8)^2(1 - xy)^2 + \sqrt{x^2 - y^2} = 0.$$

6. Does there exist a positive integer n such that the factorial $n!$ has exactly 154 zeroes at the end? (The *factorial* $n!$ is the product $1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$.)

Round One, Open Division

1. In the figure, the radius of the big circle is 6, the small circles are of equal size, and the outermost and innermost circles are tangent to the other circles. Determine the area of the shaded part of the figure.



2. Solve the Diophantine equation

$$x^2 + (10y - y^2)^2 + y^6 = 2011,$$

i.e. find the integer solutions of the equation.

3. Let $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \frac{x^2 - 2011x + 1}{x^2 + 1}.$$

Show that $|f(x) - f(y)| \leq 2011$ for all real numbers x and y.

4. The plane is tiled with black and white unit squares in such a way that adjoining tiles either have a side or a vertex in common. A line segement in the plane is white, if there exist white squres such that the segment is completely inside them, if the points of intersection of the segment and the sides of the squares are neglected. A black line segment is defined in an analogous manner. Show that the plane can be tiled in such a way that no white or black segment is longer than 5.

Round Two

1. A chord divides a circle into two segments. A square is inscribed in each of them in such a way that two vertices of the square are on the chord and two on the circumference. The ratio of the sidelengths of the squares is $5 : 9$. Compute the the ratio of the length of the chord to the radius of the circle.

2. Assume $x \neq 1$, $y \neq 1$ and $x \neq y$. Show that if

$$\frac{yz - x^2}{1-x} = \frac{zx - y^2}{1-y},$$

then

$$\frac{yz - x^2}{1-x} = \frac{zx - y^2}{1-y} = x + y + z.$$

3. Prove that the number $k^{k-1} - 1$ is divisible by the number $(k-1)^2$, for every integer $k \geq 2$.

4. Let k, $n \in \mathbb{N}$, $0 < k \leq n$. Prove that

$$\sum_{j=1}^{k} \binom{n}{j} = \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k} \leq n^k.$$

5. The *Collatz function* is a mapping $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that

$$f(x) = \begin{cases} 3x + 1 & \text{for odd } x, \\ x/2 & \text{for even } x. \end{cases}$$

We denote $f^1 = f$ and, inductively, $f^{k+1} = f \circ f^k$, i.e. $f^k(x) = \underbrace{f(\ldots(f}_{k \text{ times}}(x)\ldots)$.

Prove that there is an $x \in \mathbb{Z}_+$ such that

$$f^{40}(x) > 2012x.$$

Solutions

Round One, Basic Division

1. $5^{140} \cdot 8^{47} = 5^{140} \cdot 2^{141} = 2 \cdot 10^{140}$. There are 141 digits, an odd number. So a) and d).

2. The total area of the figure is 9. Denote by P the intersection of AB and the horizontal through E. The areas of the right triangles APE, BPE and CPE are $2h$, where h is AP, BP or CP. The hypotenuse of any of these troiangles divides the fiqure in two parts of equal area if and only if $1 + 2h = \frac{9}{2}$ or $h = \frac{7}{4}$. So b) is correct, a) and c) not. Clearly d) is even more off the mark than c).

3. A hexagon has at most $\frac{1}{2} \cdot 6(6-3) = 9$ diagonals. So a) is correct. Assuming a point P common to all diagonals of $ABCDEF$, the lines AC, AD, AF all have two points in common, which means that A, C, D, E are collinear. In a similar fashion one gets that all vertices of the hexagon are collinear. So b) is incorrect. c) and d) clearly are correct.

4. By (the inverse) of Pythagoras, the triangle is a right triangle. Statements b) and c) are correct. Putting, say $a = 2$ and $a = 3$ one sees that d) is correct.

5. a) is correct. Clearly $f(3) + f(-3) = 4$, and $f(-3) = -1$. So c) is correct, b) and d) are not.

6. A number with all digits different can have at most 10 digits. Now $100^5 = 10^{10}$ has 11 digits. There are at most 99 numbers of the kind desired. Assuming $64 \le n < 100$, we have $10^{10} > n^5 > 64^5 = 2^{30} > 1000^3 = 10^9$, so n^5 has 10 digits. If there are no repeated digits, the sum of digits is 45. So n^5 and n have to be multiplies of 3. There are 12 such numbers in the interval [64, 99] so the number of desired numbers is at most $63 + 12 = 75$. Among these, all 7 multiplies of 10 have to be counted out: their fifth powers have five zeros. So at most 68 numbers of the desired kind remain. So only c) is correct. [Actually, there are exactly 10 numbers of the desired type. They are 1, 2, 3, 4, 5, 7, 8, 14, 16 and 38 with fifth powers 1, 32, 243, 1024, 3125, 16807, 32768, 537824, 1048576 and 79235168, respectively.]

7. When the trains move in the same direction, the faster train moves with relative velocity $u - v$ compared to the slower one. When they move in the opposite direction, the relative velocity is $u + v$. The passing times are

$$t_1 = \frac{2a}{u-v}, \quad t_2 = \frac{2a}{u+v}.$$

From $t_1 = 2t_2$ one easily solves $\frac{u}{v} = 3$.

8. The inequality is true for $x \le 0$. For $0 < x < 1$, $x^6 - x^3 + x^2 - x + 1 = x^6 + (1-x)(x^2+1) > 0$. For $x \ge 1$, $x^6 - x^3 + x^2 - x + 1 = x^3(x^3-1) + x(x-1) + 1 \ge 1$.

Round One, Intermediate Division

1. We may assume that the total money involved is $270x$. At the start the playres have $108x$, $90x$ and $72x$, at the end $105x$, $90x$ and $75x$. So the last player is the only winner, and $3x = 3$. So $x = 1$, and b) is the correct answer.

2. f, $f(x) = x^{2011} + x + 1$ is strictly increasing, and has exactly one zero. a) is correct. $f(0) > 0$, so this zero is negative. c) is not correct. $f(-1) = -1$, which means that the zero is in the interval $(-1, 0)$ d) is correct. The well-known criterion on the coefficients of a polynomial with a rational zero leaves only ± 1 as possible roots of $f(x) = 0$. So b) is incorrect.

3. A routine check shows that 211 is indeed a prime. So a) is correct and b) incorrect. An odd number is the sum of two primes only if one of the primes is 2. But $209 = 19 \cdot 11$ is not a prime. So c) is incorrect. But 211 can be written as a sum of three primes in several ways, for instance $211 = 101 + 97 + 13$. d) is correct.

4. The altitude of triangle k_1 is $\dfrac{a}{\sqrt{2}}$ whence the side of k_1 is $\dfrac{2}{\sqrt{3}}\dfrac{a}{\sqrt{2}} = \dfrac{2a}{\sqrt{6}}$. The altitude and half of the side s_2 of k_2 together make the leg of the right triangle, or $\left(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\right) s_2 = a$. Solving, we obtain

$$\frac{s_1}{s_2} = \frac{1+\sqrt{3}}{\sqrt{6}}.$$

5. We have to have $x^2 - y^2 = 0$ and either $x^2 + y^2 - 8 = 0$ or $1 - xy = 0$. So $|x| = |y|$ and either $2x^2 = 8$, $|x| = 2$, or $|x| = 1$, $x = y$. The six solutions are $(1, 1)$, $(-1, -1)$, $(2, 2)$, $(2, -2)$, $(-2, 2)$, $(-2, -2)$.

6. The number of final zeroes in $n!$ equals the exponent of 5 in the prime factor decomposition of $n!$. Now if $n \ge 625$, this exponent is at least $125 + 25 + 5 + 1 = 156$, while for $n < 625$ this exponent does not exceed $156 - 4 = 152$. So there is no such n.

Round One, Open Division

1. The small circles have radius 2. The area of the shaded region is $1/6$ of the area 36π of the big circle from which 7 areas 4π of the small circles and 6 areas of the regions between the small circles are deducted. Since the lines joining the centers of the small circles pass through the points of tangency, the latter areas are equal to the area of an equilateral triangle of side 4 from which three 60° sectors of a circle of radius 2 have been deducted. Summing all this, the area of the shaded region in

$$\frac{1}{6}\left(36\pi - 28\pi - 6\cdot(4\sqrt{3} - 2\pi)\right) = \frac{10}{3}\pi - 4\sqrt{3}.$$

2. Clearly $y^6 < 2011$ is a necessary condition for solution. Since $4^6 = 2^{12} > 4000$, this means $|y| \le 3$. Also, $2011 = x^2 + (10y - y^2)^2 + (y^3)^2$, a sum of three squares. Since $2011 \equiv 3 \mod 3$ and squares are either 0 or 1 $\mod 4$, all the three squares, and hence y, are odd. Putting $y = -3, -1, 1$ lead to quadratic equations for x with non-integer solutions. But $y = 3$ gives $x^2 = 841$, $x = \pm 29$.

3. Clearly $\lim_{x\to\pm\infty} = 1$. Since

$$f'(x) = \frac{2011(x^2-1)}{(x^2+1)^2},$$

f is increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 1)$. As $f(-1) = \dfrac{2013}{2} > 1$ and $f(-1) = -\dfrac{2009}{2} < 1$, $f(-1)$ is the largest and $f(1)$ is the smallest value assumed by f. So $|f(x) - f(y)| \le f(-1) - f(1) = 2011$ for any x, y.

4. Put the squares on the plane so that their sides are parallel to the coordinate axes and centers coincide with points of integer coordinates. Put a white square on (m, n) if m is a multiple of three and n is even or if m is not a multiple of 3 and n is odd. So the tiling consists of unit squares and 2×1 rectangles. The only way a white or black segment can cross the boundaries of a white or black rectangle is by passing through the vertices. If the segment crsses only one line $y = m + \dfrac{1}{2}$, it will stay inside three unis square, and its length does not exceed $3\sqrt{2} < 5$. If it crosses two such lines, its inclination is either 1 or $\dfrac{1}{2}$. In the former case the length of the segment is at most $3\sqrt{2}$, in the latter $2\sqrt{5} < 5$.

Round Two

1. Letting a and b, $9a = 5b$, be the sides of the squares, r the rdius of the circle, s the length of the chord, and d the distance of the center of the circle from the segment, one easily finds right triangles from which the equations

$$\begin{cases} (a+d)^2 + \left(\dfrac{a}{2}\right)^2 = r^2, \\ (b-d)^2 + \left(\dfrac{b}{2}\right)^2 = r^2, \\ d^2 + \left(\dfrac{s}{2}\right)^2 = r^2 \end{cases}$$

can be read. The two first equations yield $8d = 5(b-a)$, and $d = \dfrac{a}{2}$. The first equation now gives $r = \dfrac{a\sqrt{10}}{2}$. The third equation now leads to $s = 3a$, and $s : r = 3\sqrt{10} : 5$.

2. Set
$$\lambda = \frac{yz - x^2}{1-x} = \frac{xz - y^2}{1-y},$$
or $\lambda(1-x) = yz - x^2$ and $\lambda(1-y) = zx - y^2$. Since $x \neq y$, we have
$$\begin{aligned}\lambda = \frac{\lambda(y-x)}{y-x} &= \frac{\lambda(1-x) - \lambda(1-y)}{y-x} = \frac{yz - x^2 - zx + y^2}{y-x} \\ &= \frac{y(x+y+z) - x(x+y+z)}{y-x} = x+y+z.\end{aligned}$$

3. Since $k^{k-1} - 1 = (k-1)\sum_{j=0}^{k-2} k^j$, we have to show that the sum is a multiple on $k-1$. But there are exactly $k-1$ terms in the sum, so it will be equal to
$$(k-1) + \sum_{j=0}^{k-2}(k^j - 1).$$
But each term in the last sum is a multiple of $k-1$, and we are done.

4. Let A be a set of k elements and B a set of n elements. There are n^k different mappings $f : A \to B$. Now the image sets of these mappings are all subsets of B with at most k elements. The number of these sets is $\sum_{j=1}^{k} \binom{n}{j}$, so clearly the sum is at most n^k.

5. First show by induction on k that $f^{2k}(2^k m - 1) = 3^k m - 1$ for all $m \in \mathbb{Z}_+$. In fact, $f^2(2m-1) = f(f(2m-1)) = f(3(2m-1)+1) = f(6m-2) = 3m-1$, and assuming the claim to be true for k, then $f^{2(k+1)}(2^{k+1}m - 1) = f^2(f^{2k}(2^k(2m)-1)) = f(f(3^k(2m)-1)) = f(3(3^k(2m)-1)+1) = f(3^{k+1}(2m)-2) = 3^{k+1}m - 1$, and the claim is true for $k+1$. So for $x = 2^{20} - 1 = 1048575$ we have $f^{40}(x) = 3^{20} - 1 > \left(\frac{3}{2}\right)^{20}(2^{20}-1) > 2012x$, because
$$\left(\frac{3}{2}\right)^{20} = \left(\frac{81}{16}\right)^5 > 5^5 = 3125 > 2012.$$