Finnish High School Mathematics Contest 2012 – 2014

The Finnish High School Mathematics Contest is organized annually by MAOL, the Finnish Association of Mathematics and Science Teachers. The Finnish Mathematical Society's Training Division participates in problem selection and marking of solutions. The contest is the first stage in the Finnish IMO Team selection process. It takes place in two rounds. As the division into grades no more exists in the Finnish high school sysytem, Round One has two divisions in which the age of the contestants is limited, and one division open for all students regardless of their age or school level. Round One is organized in schools in November. Time allowed is 120 minutes. Altogether about 1500 students participate, the number rather evenly devided between the divisions. Round Two is in Helsinki in January or February. About 20 best students from all divisions in Round One are invited, most of them from the Open Division. In Round Two, the working time is 180 minutes.

In Round One, Basic and Intermediate Division, part of the problems are multiple choice. The number of correct answers to these was not restricted to one for each problem.

Problems, School Year 2012–13

Round One, November 13, 2012

Basic Division

1. A diamond was broken into two pieces with masses in the ratio 3 : 4. The value of a diamond is directly proportional to the square of its mass. The value of the broken diamond in comparison to the original

a) is less than one half
b) has diminished less than 50 %
c) is about 51 %
d) is about 49 %

2. Of the pupils of a certain school, four fifths were Huns and the rest were Vandals. Because of misbehaviours, two thirds of the Huns had to be dismissed fron the school, but it was not necessary to dismiss any Vandal. What was the proportion of Vandals in the school after the dismissals?

a) 1/3 b) 3/8 c) 3/7 d) 1/2

3. Which ones of the following comparisons are true?

a) $2^{2^2} > 3^3$
b) $3^{3^3} \geq 5^5$
c) $(-4)^{-4} < (-5)^{-5}$
d) $\left((-2)^2\right)^{(-2)^2} = 4^4$

4. The number $\dfrac{2^{2013}+2^{2011}}{2^{2012}-2^{2010}}$ equals

a) 2 b) $\dfrac{10}{3}$ c) $2^{2012}+1$ d) $\dfrac{2^{13}+2^{11}}{2^{12}-2^{10}}$

5. A hemisphere and a right cone have equal bases and their surface areas are equal. Denote by V_{pp} the volume of the hemisphere and by V_{k} the volume of the cone. Then

a) $V_{\text{pp}} > V_{\text{k}}$ b) $V_{\text{pp}} < V_{\text{k}}$ c) $V_{\text{k}}/V_{\text{pp}} = \sqrt{3}/2$ d) $V_{\text{k}}/V_{\text{pp}} = 3/\sqrt{2}$

6. A cube with edge 5 is built using several smaller cubes of edges 1, 2 and 3. Then

a) It is not necessary to use all three types of small cubes.
b) The number of small cubes needed is at least 50.
c) By examining the surface of the large cube, one cannot necessarily find out whether the number of small cubes used is less than or more than 100.
d) The total surface area of the of the small cubes used is an odd integer.

7. Two circles have radii $\sqrt{3}$ and 1, and the distance between their centres is 2. Compute the area common to the two circles.

8. Let f be a quadratic polynomial with integer coefficients. Furthermore, $f(k)$ is divisible by five when k is an integer. Show that all coefficients of the polynomial f are divisible by five.

Intermediate Division

1. Which of the following statements are true for all acute angles α?

a) $\sin^2\alpha + \tan^2\alpha = 1$ b) $\sin\alpha \leq \alpha$
c) $\tan^2\alpha + \cos^2\alpha \geq 1$ d) $\tan\alpha\cos\alpha = \sin\alpha$

2. The largest integer which divides evenly certain two positive integers is 4, and 24 is the smallest positive integer, which is divisible by both of these integers. The sum of the numbers can be

a) 20 b) 24 c) 28 d) 36

3. A solid (not hollow) cube with edge 5 is constructed using several smaller cubes with edges 1, 2 and 3. Then

a) It is not necessary to use all three types of small cubes.
b) The number of small cubes needed is at least 50.
c) By examining the surface of the large cube, one cannot necessarily find out whether the number of small cubes used is less than or more than 100.
d) The total surface area of the of the small cubes used is an odd integer.

4. Let f be a quadratic polynomial with integer coefficients. Furthermore, $f(k)$ is divisible by five when k is an integer. Show that all coefficients of the polynomial f are divisible by five.

5. AB is the diameter of a circle. A tangent to the circle is drawn from a point C on the extension of AB (outside the circle), the point of tangency is N. The bisector of angle $\angle ACN$ intersects the segments AN and NP at P and Q, respectively. Prove that $|PN| = |NQ|$.

6. Determine the largest n for which the following is possible: every square in a $n \times n$ grid can be painted red or blue so that for any two rows and any two columns, the squares at their crossings are not all of the same colour.

Open Division

1. The operation $*$ for real numbers is defined as follows: $a * b = a^2 + b^2 - ab$. Solve the equation

$$(x * x) * 1 = x * (x * 1).$$

2. Olli and Liisa play the following game with 2012 tokens. The tokens are in a heap and the players take in turn one, two or three tokens. The one who gets the last token wins. Liisa starts. Find out, whether one of the players can always guarantee a win for him/herself.

3. A cunning ruler promises half of his kingdom in return of the following work performance: for each of the 64 squares of the chessboard the worker has to toil for a certain nuber of days. For the fist square, the work will take one day and for each of the following squares, the number of working days will be two times the number of days needed for the previous square. The work will start on a Monday and it will continue without interruptions (not even on weekends). On which day of the week will the work be finished?

4. The point P is chosen on the bisector of an acute angle $\angle A$, and the point B is on one of its sides. The line BP intersects the other side at C. Show that the value of

$$\frac{1}{AB} + \frac{1}{AC}$$

is independent of the choice of B as long as P is fixed.

Round Two, February 1, 2013

1. The coefficients of the polynomial function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 + ax^2 + bx + c$, are pairwise different non-zero integers. Moreover, $f(a) = a^3$ and $f(b) = b^3$. Determine a and b.

2. In a European city, tickets for public travel are sold as 7 day and 30 day tickets. The previous ones cost 7.03 euros apiece and the latter 30 euros apiece. Al Gebra decides to obtain all the tickets he needs for his three year (2014–2016), i.e. 1096 day long stay. What is the most advantageous solution?

3. The points A, B and C lie on the circumference of the unit circle. Moreover, we know that AD is the diameter of the circle and

$$\frac{|AC|}{|CB|} = \frac{3}{4}.$$

The bisector of angle ABC intersects the circumference at D. Determine the length of the segment AD.

4. We call a subset of the set $\{1, 2, 3, \ldots, 50\}$ *special*, if it contains no pair $\{x, 3x\}$. A special subset is *superspecial*, if it contains a maximal number of elements. How many elements does a superspecial set contain and how many different superspecial sets exist?

5. Find all triples of integers (m, p, q) such that

$$2^m p^2 + 1 = q^5,$$

$m > 0$, and p and q are prime numbers.

Problems, School Year 2013–14

Round One, Novenber 12, 2013

Basic Division

1. Which of the following pairs of numbers are equal?

a) $\sqrt{2}$ and 1,414213562373
b) $\sqrt{5-2\sqrt{6}}$ and $\sqrt{2}-\sqrt{3}$
c) $\sqrt{7}$ and 2,645751311064
d) $\sqrt{9+2\sqrt{14}}$ and $\sqrt{7}+\sqrt{2}$.

2. More air is pumped into a balloon so much that the volume of the balloon increases by 237,5 %. Then the surface area of the balloon increases by

a) 100 % b) 125 % c) at least 150 % d) at most 175 %.

3. There are $n > 1$ numbers and their average is $M \neq 0$. One of the numbers, a, is removed, and the average of the remaining numbers is computed.

a) The new average is $\dfrac{M-a}{n-1}$.
b) The new average can be smaller than the original average.
c) The difference of the new average and M is $\dfrac{M-a}{n-1}$.
d) The average of the new average and M is $\dfrac{nM-a}{2(n-1)}$.

4. The expression

$$\frac{c}{a+\dfrac{b}{c}} + \frac{a+c}{a-\dfrac{b}{c}}$$

is simplified. Which of the following outcomes are correct for all values of a, b and c?

a) 0
b) $\dfrac{c(2bc+a^2c+ab)}{b^2-a^2c^2}$
c) $\dfrac{ac(2c^2+b+ac)}{a^2c^2-b^2}$
d) $\dfrac{ac}{ac-b} + \dfrac{2ac^3}{(ac+b)(ac-b)}$.

5. For a positive integer n, denote by $S(n)$ the sum of the digits of n (in base 10). Which of the following statements are true for all positive integers n?

a) $S(3n)$ is divisible by 3 b) $S(2n) \leq 2S(n)$

c) $S(2n) \geq \dfrac{1}{2}S(n)$. d) $S(7n)$ is divisible by 7.

6. We know that

$$\frac{8^x}{2^{x+y}} = 64 \quad \text{and} \quad \frac{9^{x+y}}{3^{4y}} = 243.$$

Then $2xy$ is

a) a negative number b) 5

c) 7 d) an odd integer.

7. P is a point on the hypothenuse AB of a right triangle ABC. We know that $|PB| : |PC| : |PA| = 1 : 2 : 3$. Determine the ratios of the lengths of the sides of the triangle.

8. Show that if the real numbers x, y and z satisfy $(x + y + z)^2 = 3(xy + xz + yz)$, then all the numbers are necesarily equal.

Intermediate Division

1. There are $n > 1$ numbers and their average is $M \neq 0$. One of the numbers, a, is removed, and the average of the remaining numbers is computed.

a) The new average is $\dfrac{M - a}{n - 1}$.

b) The new average can be smaller than the original average.

c) The difference of the new average and M is $\dfrac{M - a}{n - 1}$.

d) The average of the new average and M is $\dfrac{nM - a}{2(n - 1)}$.

2. For a positive integer n, denote by $S(n)$ the sum of the digits of n (in base 10). Which of the following statements are true for all positive integers n? is a positive

a) $S(3n)$ is divisible by 3 b)$S(2n) \leq 2S(n)$

c) $S(2n) \geq \frac{1}{2}S(n)$. d) $S(7n)$ is divisible by 7.

3. The equation $x^3 + 3ax^2 + bx + c = 0$ has three solutions which form an arithmetic sequence. (Three numbers are in an arithmetic sequence, if one of them is the average of the other two.) Then always

a) $ab = 2a^3 + c$ b) $a = 0$ c) $3a + c = 2b$ d) $b = 3ac$.

4. In the triangle ABC, $AB < AC$. Let S be the circumcircle of ABC. A line through A, perpendicular to BC meets S again at the point P. The point X is on the line segment AC, and the extension of BX meets the circle S at the point Q. Prove: if $|BX| = |CX|$, then PQ is the diameter of S.

5. A board solitaire is played with one blue and three white pieces, which can be placed in the squares of a 2013×2013 board. One move consists of taking one of the pieces and moving it to the left, to the right, up or down all the way until the border of the board or another piece is encountered. Prove that the blue piece can be moved to any of the squares of the board, regardless of the initial position of the pieces.

6. Find all positive integers m and n such that n is odd and the equation

$$\frac{1}{m} + \frac{4}{n} = \frac{1}{12}$$

is fulfilled.

Open Division

1. We know that

$$\frac{8^x}{2^{x+y}} = 64 \quad \text{and} \quad \frac{9^{x+y}}{3^{4y}} = 243.$$

Determine $2xy$.

2. In average, one person in a million suffers from an uncommon disease. The disease is diagnosed with a test which gives a correct result with probability 99 %, regardless of whether the person has the disease or not. An arbitrarily chosen person is tested positive, i.e. as having the disease. What is the probability that he has the disease?

3. There are two points A and B on a sheet of paper. Their distance is more than 10 cm but less than 20 cm. You have a ruler exactly 10 cm long and a pair of compasses. How can you draw the line AB using these tools?

4. For a positive integer n, denote by $S(n)$ the sum of the digits of n (in base 10). Which rational numbers q can be written as

$$q = \frac{S(2n)}{S(n)}$$

for some positive integer n?

Round Two, January 31, 2014

1. Determine the value of the expression

$$x^2 + y^2 + z^2,$$

if

$$x + y + z = 13, \quad xyz = 72 \quad \text{and} \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{4}.$$

2. The center of the circumcircle of the acute triangle ABC is M, and the circumcircle of ABM meets BC and AC at P and Q ($P \neq B$). Show that the extension of the line segment CM is perpendicular to PQ.

3. The points $P = (a, b)$ and $Q = (c, d)$ are in the first quadrant of the xy plane, and a, b, c and d are integers satisfying $a < b$, $a < c$, $b < d$ ja $c < d$. A *route* from point P to point Q is a broken line consisting of unit steps in the directions of the positive coordinate axes. An allowed route is a route not touching the line $x = y$. Tetermine the number of allowed routes.

4. The radius r of a circle with center at the origin is an odd integer. There is a point (p^m, q^n) on the circle, with p and q prime numbers and m and n positive integers. Determine r.

5. Determine the smallest number $n \in \mathbb{Z}_+$, which can be written as $n = \sum_{a \in A} a^2$, where A is a finite set of positive integers and $\sum_{a \in A} a = 2014$. In other words: what is the smallest positive number which can be written as a sum of squares of different positive integers summing to 2014?

Answers and Solutions, School Year 2012–13

Basic Division

1. a) false; b) and c) true; d) false.

2. a) and b) false; c) true; d) false.

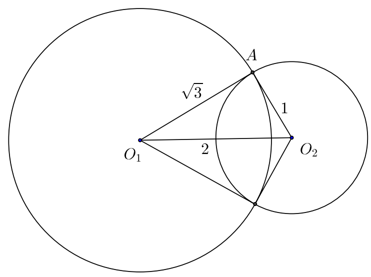
3. a) false; b) true; c) false; d) true.

4. a) false; b) true; c) false; d) true.

5. a) true; b) false; c) true; d) false.

6. a) true. b) Set the cube in an xyz-coordinate system in such a way that its center is in the origin and edges parallel to the coordinate axes. Any cube used in the assembling of the 5×5 cube with edge legth ≥ 2 contains at least one of the eight points $(\pm 1, \pm 1, \pm 1)$. And at most one of the 3×3 cubes can be used. So the volume occupied by cubes bigger than the unit cube is at most $3^3 + 7 \cdot 2^3 = 83$. At least 42 unit cubes are needed; $42 + 8 = 50$, so b) is true. c) By examining the surface we cannot know hoe the 3×3 interior is composed, so true; d) The total surface area of any number of cubes with integer edge is a multiple of 6, so false.

7. If O_1 and O_2 are the centers of the circles and A their common point, then, by Pythagoras, O_1O_2A is a right triangle and $\angle AO_1O_2 = 30°$, $\angle AO_2O_1 = 60°$. The total area covered by the circles can be computed by taking a $300°$ sector of the larger and a $240°$ sector fron the smaller plus twice the area of the right triangle O_1O_2A; the result is $\dfrac{19}{6}\pi + \sqrt{3}$. The sum of the areas of the circles is 4π, so the area common to the circles will be $\dfrac{5}{6}\pi - \sqrt{3}$.



8. Let $f(x) = ax^2 + bx + c$, a, b, $c \in \mathbb{Z}$. Then $c = f(0)$, $a+b+c = f(1)$ and $a-b+c = f(-1)$ are multiples of 5. Then $2b = f(1) - f(-1)$ and $2(a+c) = f(1) + f(-1)$ are multiples of five; 2 and five are primes, so b and $a+c$ are multiples of five, and finally a too.

Intermediate division

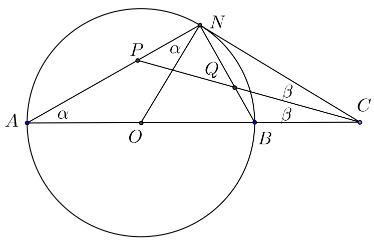
1. a) false; b), c) and d) true.

2. a) true; b) false; c) true; d) false.

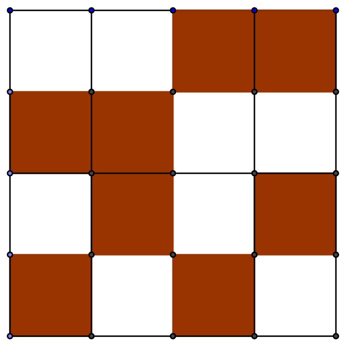
3. a), b) and c) true; d) false. See Basic Division, problem 6.

4. See Basic Division, problem 8.

5. Let O be the center of the circle and let $\angle NAB = \alpha$, $\angle NCP = \angle PCA = \beta$. Then $\angle ANO = \alpha$ and since both $\angle ANB$ and $\angle ONC$ are right angles, $\angle BNC = \angle ANO = \alpha$. Looking at triangles ACP and CNQ we see that $\angle NPQ = \angle NPC = \alpha+\beta$ as well as $\angle NQP = \alpha + \beta$. SO $\angle NPQ = \angle NQP$, and the triangle NPQ is isosceles.



6. If $n = 4$, the coloring in the figure shows that the required coloring is possible. If $n = 5$, there are at least three squares of the same colour in the uppermost row; assume they are red. Looking at the columns of these red squares, if on any row below the uppermost row there are two red squares, a rectangle with red corners appears. So each of these rows has at lest two blue squares. There are at most three ways these two blue squares can appear on the three columns. Since there are four rows, the blue squares are on identical positions on at least two rows. These four blue squares are then corners of a rectangle. So no 5×5 square fulfills the condition, and consequently no $n \times n$ square for $n \geq 5$.



Open Division

1. According to the definition of $*$, $(x * x) * 1 = x * (x * 1)$ which implies $(x^2) * 1 = x * (x^2 - x + 1)$ and further to $x^4 - 2x^2 + 1 = (x^2 - 2x + 1)(x^2 - x + 1)$. This simplifies to $(x^2 - 1)^2 = (x - 1)^2(x^2 - x + 1)$. So either $x = 1$ or $(x + 1)^2 = x^2 - x + 1$. The quadratic equation reduces to $3x = 0$. So $x = 0$ and $x = 3$ are only possible solutions. That they indeed are solutions, is easy to check.

2. The player who can make a move when only 1, 2 or three tokens are left, wins. The player who moves when four tokens are left, can only leave 1, 2, or 3 tokens to the opponent. Since 2012 is a multiple of 4, Olli, the beginner, can always reach a number of tokens divisible by 4, and win.

3. The number of days needed for three consecutive squares is $2^k + 2^{k+1} + 2^{k+2} = 7 \cdot 2^k$. The work needed for square one will be done on Monday, and the rest $63 = 21 \cdot 3$ squares can be divided in three square sequencec each taking a whole number of weeks. So the work will be finished on a Monday.

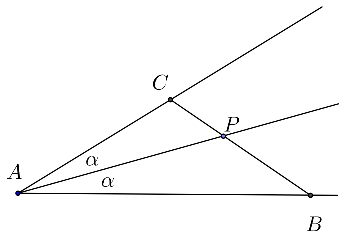
4. Let $\angle A = 2\alpha$. Computing the area on ABC in two ways, we get

$$\frac{1}{2}AB \cdot AC \sin(2\alpha) = \frac{1}{2}AP \cdot AC \sin\alpha + \frac{1}{2}AB \cdot AP \sin\alpha,$$

which immediately implies

$$\frac{1}{AP}\frac{\sin(2\alpha)}{\sin\alpha} = \frac{1}{AB} + \frac{1}{AC}.$$

Since α and AP are constants, so is the right hand side of the equation.



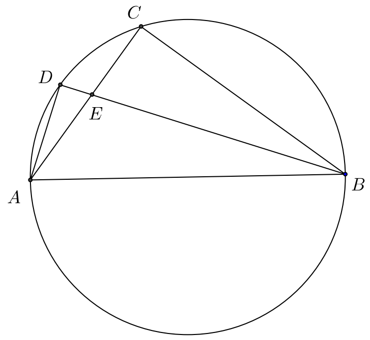
Round Two

1. Let $g(x) = f(x) - x^3 = ax^2 + bx + c$. We know that $g(a) = g(b) = 0$. So $g(x) = a(x-a)(x-b)$. The integers a, b and c must satisfy the equations $b = -a(a+b)$ and $c = a^2 b$. The former implies

$$b = -\frac{a^2}{a+1} = -\frac{a^2 - 1 + 1}{a+1} = 1 - a - \frac{1}{a+1}.$$

Since b has to be an integer, we must have $a + 1 = \pm 1$. Since $a \neq 0$, only $a = -2$ is possible. So necessarily $b = 4$ and $c = 16$.

2. We assume that Al Gebra buys m seven day tickets and n 30 day tickets. To make the tickets suffice, one has to have $7m + 30n \geq 1096$. The tickets cost $m \cdot 7{,}03 + n \cdot 30 = 0{,}03 \cdot m + 7m + 30n \geq 0{,}03m + 1096$. We determine m and n such that $7m + 30n = 1096$. The numbers 7 and 30 are relatively prime. So we can solve the Diophantine equation $7m + 30n = 1$ by the Euclidean algorithm: since $30 = 4 \cdot 7 + 2$ and $7 = 3 \cdot 2 + 1$, we have $1 = 7 - 3 \cdot 2 = 7 - 3 \cdot (30 - 4 \cdot 7) = 13 \cdot 7 - 3 \cdot 30$. So $13 \cdot 1096 \cdot 7 - 3 \cdot 1096 \cdot 30 = 1096$ and $(13 \cdot 1096 - 30t) \cdot 7 - (3 \cdot 1096 - 7t) \cdot 30 = 1096$ for all t. To get the smallest possible m we must take t as the largest integer for which $13 \cdot 1096 - 30t \geq 0$ and $-(3 \cdot 1096 - 7t) \geq 0$. The largest t, for which $13 \cdot 1096 \geq 30t$ is 474; then $m = 28$. We see that $3 \cdot 1096 - 7 \cdot 474 = -30 < 0$. So if Al Gebra buys 28 seven day tickets and 30 30 day tickets, he has to pay $1096 + 28 \cdot 0{,}03 = 1096{,}84$. For any other choice of m and n $0{,}03m + 7m + 30n \geq 0{,}03m + 1097 \geq 1097$.



3. Since AB is the diameter of the unit circle, $AB = 2$. To facilitate the computation we consider a circle with diameter $A'B' = 15$ (we use dotted letters for points in this "magnification"). Because $A'B'$ is a diameter, $\angle A'C'B' = 90°$. The legs of the right triangle $A'B'C'$ are in relation $3 : 4$. So the triangle is the well-known $(3 : 4 : 5)$ right triangle, and $A'C' = 9$, $B'C' = 12$. An angle bisector divides the opposite side in the ratio of the adjacent sides. If $B'D'$ meets $A'C'$ at E', then $A'E' = 5$ and $E'C' = 4$. $E'B'C'$ is a right triangle, so $B'E'^2 = 12^2 + 4^2 = 160$ and $B'E' = 4\sqrt{10}$. The angles $\angle D'A'C'$ and $\angle D'B'C'$ subtend the same arc and are equal. The triangles $A'E'D'$ and $B'E'C'$ are similar. So

$$\frac{A'D'}{A'E'} = \frac{B'C'}{B'E'}$$

or

$$A'D' = 5 \cdot \frac{12}{4\sqrt{10}} = \frac{15}{\sqrt{10}}.$$

As our original circle had diameter 2, we have to multiply $A'D'$ by $\frac{2}{15}$; the answer to the problem thus is $AD = \frac{2}{\sqrt{10}} = \frac{\sqrt{10}}{5}$.

4. A special subset cannot have two consequtive elements from the sets $E_1 = \{1, 3, 9, 27\}$, $E_2 = \{2, 6, 18\}$, $E_3 = \{4, 12, 36\}$, $E_4 = \{5, 15, 45\}$, $E_5 = \{7, 21\}$, $E_6 = \{8, 24\}$, $E_7 = \{10, 30\}$, $E_8 = \{11, 33\}$, $E_9 = \{13, 39\}$, $E_{10} = \{14, 42\}$, $E_{11} = \{16, 48\}$, $E_{12} = \{17\}$, $E_{13} = \{19\}$, $E_{14} = \{20\}$, $\ldots$, $E_{34} = \{50\}$ ("$\ldots$" refers to all sets $\{k\}$, where $22 \leq k \leq 49$ and k is not a multiple of three). A superspecial subset has two elements from the sets $E_1, \ldots E_4$ and one element from each of the sets $E_5, \ldots, E_{34}$. A superspecial subset thus has $8 + 30 = 38$ elements. From set E_1 the selection can be made in three different ways, from E_2, E_3, E_4 and $E_{12} \ldots E_{34}$ in one way only, while from $E_5, \ldots, E_{11}$ the selection can be made in two ways. So there are $3 \cdot 2^7 = 384$ different superspecial subsets.

5. Numbers satisfying the condition of the problem also satisfy

$$2^m p^2 = (q-1)(q^4 + q^3 + q^2 + q + 1).$$

Since the second factor on the right hand side is odd and > 1, necessarily $q - 1 = 2^m$ or $q - 1 = 2^m p$. The latter equation would imply $2^m p^2 + 1 = (2^m p + 1)^5 > 2^{5m} p^5$, which clearly is impossible. So $q = 2^m + 1$. But now $2^m p^2 + 1 = (2^m + 1)^5 = 2^{5m} + 5 \cdot 2^{4m} + 10 \cdot 2^{3m} + 10 \cdot 2^{2m} + 5 \cdot 2^m + 1$ or

$$p^2 = 2^{4m} + 5 \cdot 2^{3m} + 10 \cdot 2^{2m} + 10 \cdot 2^m + 5.$$

If $m \geq 2$, we obtain $p^2 = 8k + 5$. But the quadratic rests of squares $mod 8$ are 0, 1 or 4. So $m < 2$ or $m = 1$. Then $q = 3$ and $2p^2 = 3^5 - 1 = 242$. We get $p^2 = 121$ and $p = 11$.

Answers and Solutions, School Year 2013–14

Basic Division

1. a), b) and c) are obviously false; since $(\sqrt{7}+\sqrt{2})^2 = 7 + 2\sqrt{7\cdot 2} + 2 = 9 + 2\sqrt{14}$, d) is true.

2. Observe that 237.5% equals $\frac{19}{8}$. Denoting the radius of the ball before inflating by r and after inflating by R, we have

$$\frac{R^3 - r^3}{r^3} = \frac{19}{8}$$

whence $\frac{R^3}{r^3} = \frac{27}{8}$ and $\frac{R^2}{r^2} = \frac{9}{4}$. So

$$\frac{R^2 - r^2}{r^2} = \frac{5}{4} = 125\%$$

. So b) and d) are true, a) and c) false.

3. When a is removed, the new average is $\frac{nM - a}{n-1}$. So a) is false. Removing e.g. the largest number shows that b) is true. Easy computations show that c) is true but d) false.

4. Setting e.g. $a = 2$, $b = c = 1$ one sees that a) and b) are false. Dull but easy computations show that c) and d) are true.

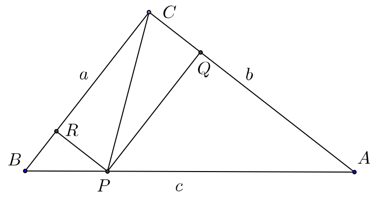
5. a) true. b) Let a be the k^{th} digit in n. If $a \leq 4$, the k^{th} there is no carry digit, and the net contribution of a to $S(2n)$ is $2a$. If $5 \leq a$, there is a carry digit, and the net contribution of a to $S(2n)$ is $2a - 10 + 1 < 2a$. Since the carry digits do not cascade, $S(2n) \leq 2S(n)$, and b) is true. c) and d) are false; counterexamples are $n = 5$ and $n = 2$, respectively.

6. The system reduces to

$$\begin{cases} 2x - y = 6 \\ 2x - 2y = 5 \end{cases}$$

with solution $2x = 7$, $y = 1$. So a) and b) are false, c) and d) true.

7. Denote the sides of ABC by a, b, c and the feet of the perpendiculars from P to AC ans BC by Q and R, respectively. The conditions of the problem then easily give $BP = \frac{1}{4}c$, $PR = \frac{1}{4}b$, $PQ = \frac{3}{4}a$ and $PC = 2 \cdot BP = \frac{1}{2}c$. Using Pythagoras to PQC then dives $9a^2+b^2 = 4c^2$, and this combined to $a^2+b^2 = c^2$ leads to $8a^2 = 3b^2$ and $5a^2 = 3b^2$. So $a : b : c = \sqrt{3} : \sqrt{5} : \sqrt{8}$.



8. The equation is equivalent to

$$(x-y)^2 + (y-z)^2 + (z-x)^2 = 0.$$

Intermediate Division

1. See problem 3, Basic Disision.

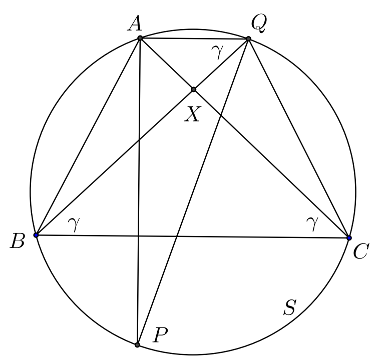
2. See problem 5, Basic Division.

3. By assumption the roots of the equation are $u-d$, u, $u+d$ for some u and $d \neq 0$. When these are substituted into the equation, the system

$$\begin{cases} (u-d)^3 + 3a(u-d)^2 + b(u-d) + c = 0 \\ u^3 + 3au^2 + bu + c = 0 \\ (u+d)^3 + 3a(u+d)^2 + b(u+d) + c = 0 \end{cases}$$

arises. Adding the first and last equation and taking the second one into account leaves $6d^2(u+a) = 0$ or $u = -a$. Substituting this into the original equation leaves $2a^3 - ab + c = 0$, which is equivalent to a). In the equation $x(x-1)(x-2) = x^3 - 3x^2 + 2x = 0$, whose solutions form the arithmetic triple $(0, 1, 2)$, none of the conditions b), c) and d) are true.

4. As base angles of the isosceles triangle XBC, $\angle ACB = \angle QBC$. As angles subtending the same arc $\overset{\frown}{AB}$, $\angle AQB = \angle ACB$. So $\angle AQB = \angle QBC$ and $AQ \| BC$. Since $AP \perp BC$, also $AP \perp AQ$. By Thales's theorem, PQ is a diameter of S.



5. Denote the squares by the coordinates (x, y), $0 \leq x, y \leq 2012$, of their centers.

That the required sequence of moves exists can be shown by examining three subprocesses. (i) It is possible to move all four pieces to form a square at the lower left corner. (ii) It is possible to rotate the pieces in this square. (iii) it is possible to move the square by one step upwards or to the right as long as the border of the board is not obstructing. To see that (i) is possible, one can first take the piece at (x_0, y_0) with smallest x, and if there are several pieces with same x, the one with smallest y, and move $(x_0, y_0) \to (0, y_0) \to (0, 0)$. Then, for the next piece (x_1, y_1) chosen with the same minimality conditions we can move $(x_1, y_1) \to (x_1, 0) \to (1, 0)$. For the remaining two pieces, at least for one the moves $(x_2, y_2) \to (x_2, 2012) \to (0, 2012)$ are possible. For the last one, the moves $(x_4, y_4) \to (x_4, 2012) \to (2012, 2012) \to (1, 2012) \to (1, 1)$ are always possible. The move $(0, 2012) \to (0, 1)$ completes the subprocess.

The rotation counterclockwise will result e.g. from the following moves: $(0, 1) \to (0, 2012)$, $(1, 0) \to (2012, 0) \to (2012, 2012) \to (1, 2012)$, $(0, 0) \to (2012, 0)$, $(0, 2012) \to (0, 0)$, $(2012, 0) \to (1, 0)$, $(1, 1) \to (0, 1)$ and $(1, 2012) \to (1, 1)$. So repeating this sequence if needed, the blue piece can be brought to any of the four places in the square.

By symmetry, we only need to show that a square of four pieces can be moved one step to the right. Assume the pieces are at (x, y), $(x+1, y)$, $(x, y+1)$, $(x+1, y+1)$. Also assume first that $y > 0$. Now move $(x+1, y+1) \to (2012, y+1)$, $(x, y) \to (x, 0) \to (2012, 0) \to (2012, y)$, then $(2012, y+1) \to (x+1, y+1)$, $(x, y+1) \to (x, 2012) \to (2012, 2012) \to (2012, y+1) \to (x+2, y+1)$ and $(2012, y) \to (x+2, y)$. In case $y = 0$, one can move $(x, 1) \to (x, 2012) \to (2012, 2012) \to (2012, 0)$, $(x, 0) \to (x, 2012) \to (2012, 2012) \to (2012, 1) \to (x+2, 1)$ and $(2012, 0) \to (x+2, 0)$.

So the four piece square can be moved so that an arbitrary square of the board is covered. The possibility to rotate the pieces ensures that an arbitrary square can be covered by just the blue piece.

6. The equation of the problem is equivalent to

$$(m-12)(n-48) = 576 = 9 \cdot 64.$$

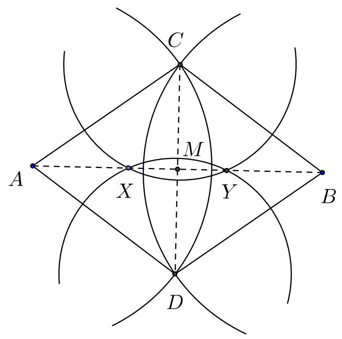
Since $n - 48$ is odd, it has to divide 9. So $|n - 48 \leq 9$ and $|m - 12| \geq 64$. Because $m > 0$, $m - 12 > 0$ and consequently $n - 48 > 0$. The three possible values 1, 3, and 9 of $n - 48$ result in three solutions (m, n): $(588, 49)$, $(204, 51)$ and $(76, 57)$.

Open Division

1. See problem 6, Basic Division.

2. Denote by p the probability that a randomly chosen person has the disease and by ϵ the probability of a test error. So $p = 10^{-6}$ and $\epsilon = 10^{-2}$. Now the probability that a person with a positive test result has the disease is

$$\frac{p(1-\epsilon)}{p(1-\epsilon) + (1-p)\epsilon} = \frac{99}{99 + 10^6 - 1} \approx 10^{-4}.$$



3. Draw circles with radius 10 cm and centers A and B. They will meet at points C D. The line CD is the perpendicular bisector of AB and AB is the perpendicular bisector of CD. It will meet AB at the latter's midpoint M. Then $CM = DM < AC < 10$. Draw circles of equal radius r, $CM < r < 10$, centered at C and D. Their common points X and Y are on the perpendicular bisector of CD, i.e. on AB. One of the points, say X, is on the segment AM. So $AX < 10$, and the line segment AX can be drawn with the short straighegde. It can be extended with the same tool indefinitely, and at least to B.
5

4. (See also problem 5, Basic Division.) If

$$n = \sum_{i=0}^{k} a_i 10^i,$$

then

$$S(n) = \sum_{i=0}^{k} a_i$$

and

$$S(2n) = \sum_{i=0}^{k} b_i,$$

where $b_i = 2a_i$, if $0 \le a_i \le 4$ and $b_i = 2a_i - 9$, if $5 \le a_i \le 9$. If $5 \le a_i \le 9$, then

$$\frac{1}{5} = 2 - \frac{9}{5} \le 2 - \frac{9}{a_i} = \frac{2a_i - 9}{a_i} = \frac{b_i}{a_i} \le 2 - \frac{9}{9} = 1.$$

Combining this with $S(2n) \le 2S(n)$, we see that

$$\frac{1}{5} \le \frac{S(2n)}{S(n)} \le 2.$$

We know prove that every rational number $\frac{p}{q}$ such that $\frac{1}{5} \le \frac{p}{q} \le 2$ equals $\frac{S(2n)}{S(n)}$ for some integer n. Consider a number n written with v 5's and u 1's, $v \ge 0$, $u \ge 0$, $v+u > 0$. Then $2n$ will be written with v 1's, a zero and u 2's. So $S(n) = 5v + u$ and $S(2n) = v + 2u$. We try to determine v and u in such a manner that

$$\frac{v + 2u}{5v + u} = \frac{p}{q}$$

or

$$(5p-q)v = (2q-p)u.$$

Since $5p-q \geq 0$ and $2q-p \geq 0$, and at least one of these inequalities is proper, one can choose $v = 2q-p$ and $u = 5p-q$.

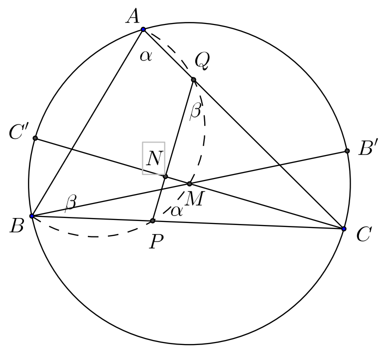
Round Two

1. Using the conditions given, we obtain

$$169 = (x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx)$$
$$= x^2+y^2+z^2+2xyz\left(\frac{1}{z}+\frac{1}{x}+\frac{1}{y}\right) = x^2+y^2+z^2+144\cdot\frac{3}{4} = x^2+y^2+z^2+96,$$

joten $x^2+y^2+z^2 = 169-108 = 61$.

2. Let BB' and CC' be diameters of the circumcle of ABC. Let CC' and QP meet at N. The claim will be proved, if $\angle CNP = \angle CNQ$. This is the case, if $\angle NPC + \angle PCN = \angle NQC + \angle NCQ$. Since $ABPQ$ is an inscribed quadrilateral, $\angle NPC = \angle QPC = \angle BAQ = \angle BAC$. The arcs subtending the angles $\angle BAC$ and $\angle C'CB = \angle NCP$ sum to the half circle arc $\overset{\frown}{C'BC}$. In a similar way we see that the arcs subtending the angles $\angle ABC$ and $\angle C'CA = \angle NCQ$ sum tu the half circle arc $\overset{\frown}{CAC'}$. But this implies $\angle NPC + \angle PCN = \angle NQC + \angle NCQ$, and the proof is complete.

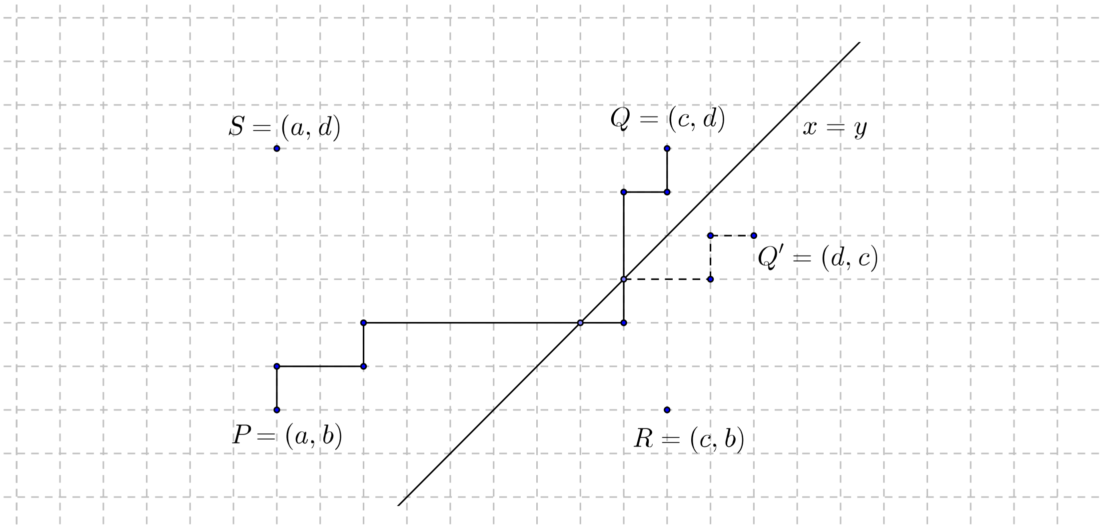


3. Let $R = (c,\, b)$ and $S = (a,\, d)$. All routes from P to Q run inside the rectangle $PRQS$. If we ignore the condition dealing with the line $x = y$, a route has $n = c-a+(d-b)$ steps, and exactly $k = d-b$ will be taken upwards, i.e. in the direction of the positive y axis. The position of the upward steps in the route determines it uniquely. So there are as many routes as there are subsets of k elements in a set of n elements, or

$$\binom{n}{k} = \binom{c-a+d-b}{d-b}.$$

If $c < b$, then $PRQS$ is totally above the line $x = y$, and all routes are allowed. Assume then $b \leq c$. Then every not allowed of *forbidden* route includes one or more points on the line $x = y$. Let $X = (t,\, t)$ be the one among these with maximal t. If the part of the foirbidden route from X to Q is reflected in the line $x = y$, and the rest of the route is left as it is, a route from P to the point $Q' = (d,\, c)$ is obtained. Because P and Q' are on different sides of the line $x = y$, every route from P to Q' is obtained by making the transformation described above to some forbidden route from P to Q. In fact the forbidden routes and routes from P to Q' are in a one to one correspondence. Let $n' = d - a + c - b$ be the length of a route from P to Q' and $k' = c - b$ the number of upward steps in these routes. The number of allowed routes then is

$$\binom{n}{k} - \binom{n'}{k'} = \binom{c-a+d-b}{d-b} - \binom{d-a+c-b}{c-b}.$$



4. Because (p^m, q^n) is on the circle $x^2 + y^2 = r^2$,

$$p^{2m} + q^{2n} = r^2. \tag{1}$$

Since r is odd, exactly one of p and q is even. Assume first that p is even and q odd. Then $p = 2$, and equation (1) yields

$$2^{2m} = r^2 - (q^n)^2 = (r + q^n)(r - q^n).$$

So

$$\begin{cases} r + q^n = 2^u, \\ r - q^n = 2^v, \end{cases} \tag{2}$$

where $u > v$ and $u + v = 2m$. The equation (2) imply $2q^n = 2^u - 2^v$. This is possible only if $v \geq 1$. Then $q^n = 2^{u-1} - 2^{v-1}$. Since q is odd, $v = 1$ and $u = 2m - 1$. This implies $q^n = 2^{2m-2} - 1 = (2^{m-1} + 1)(2^{2m-1} - 1)$. Now two consequtive odd numbers are coprime. This means that $2^{m-1} - 1 = 1$ or $m = 2$. The $q^n = 2^{2\cdot 2-2} - 1 = 3$. This is possible only if $n = 1$. So $q = 3$ and $r^2 = 2^4 + 3^2 = 25$ or $r = 5$. – If q is even, then $q = 2$, and the solution is as above.

5. If the positive numbers a, b, c satisfy $a = b + c$, then $a^2 = b^2 + c^2 + 2bc > b^2 + c^2$. So the smallest number in any set minimizing the sum of squares of the problem is 1 or 2. If $a+2 < b$, then $a+1 < b-1$; $(a+1)^2+(b-1)^2 = a^2+b^2+2(a-b)+2 < a^2+b^2$. This means that in a minimal sum, with terms in increasing order, the difference of two consequtive terms is at most 2, and this difference is 2 in at most one position of the sequence. Now

$$\sum_{k=1}^{63} k = 32 \cdot 63 = 2016.$$

The only set satisfying the minimal sum criteria presented above and summing to 2014, is $A = \{1, 3, 4, \ldots, 63\}$. Using the well known formula for the sum of squares consequtive integers we get

$$\sum_{a \in A} a = \frac{63 \cdot 64 \cdot 127}{6} - 2^2 = 85340.$$